# Stability in Matching Markets with Sizes*

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**Abstract**

Matching markets such as day care, student exchange, refugee resettlement, and couples problems involve agents of different *sizes*, that is agents who require different amounts of capacity. I study a matching market between agents and objects where the size of an agent is either one or two. Contrary to canonical models, the set of stable matchings may be empty. I identify a trade-off for existence: it is always possible to either bound the instability to a certain number of units per object or to eliminate waste but the existence of a matching that does both is not guaranteed. I develop two fairness criteria that lie on either side of this trade-off: unit-stability bounds the instability and size-stability eliminates waste. I show that size-stability is more desirable than unit-stability from a welfare point of view.

**Keywords:** stable matching, sizes, existence, unit-stable, size-stable, day care.
**JEL Classifications:** C62, C78, D47, D61, D63.

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# 1 Introduction

Centralized matching programs have been successfully implemented in various markets, including the National Resident Matching Program (Roth, 2003), school choice (Abdulkadiroglu and Sönmez, 2003), and kidney exchange (Roth, Sönmez, and Ünver, 2004). At first sight, day care appears to be an ideal market for centralized matching: children (or in fact their parents) have preferences over day care centers while day care centers have capacity constraints and rank children according to priorities.[1]

An important difference between the matching of children to day care centers and students to schools is that children often attend day care part-time while students always attend school full-time.[2] A place at a day care center can therefore be shared among two or more children attending on different days. Mathematically, children have different *sizes* in the sense that they affect the capacity constraint of a day care center differently depending on whether they attend part-time or full-time.[3]

Matching markets with sizes arise in a variety of contexts. Student exchange agreements between universities, for example those created by the Erasmus program, often contain this feature. These bilateral agreements allow students from one university to study at the other for a year or a semester. The home university is responsible for selecting which students it will send to its partner. Places are often competitive and students may apply for several destinations, in order of preference. The selection process is a matching market: students have preferences over partner universities and have different priorities for each of them depending on the quality of their application. Capacities are determined by the exchange agreement, which can specify either a maximum number of students that can be selected or a maximum number of semesters that can be used. In the latter case, a student going for a year uses two units of capacity.

Agents with different sizes also present challenges in existing models. The National Resident Matching Program (NRMP) has matched medical graduates to residency hospitals in the United States since 1951. Since 1983, doctors have had the option to apply as couples

[1]For example, families from a disadvantaged background or families who live in the neighborhood where the day care center is located often have a higher priority.

[2]Another potential difficulty with day care matching is that children may enter or exit at any time, giving the problem a dynamic aspect. The largest intake, however, tends to take place once a year when the older children start school or kindergarten. Optimizing this static problem has the potential to greatly improve the way the market operates.

[3]Capacity constraints can of course vary from one country to another. This example was inspired by the situation in Australia but Kamada and Kojima (2018) report a different problem in Japan. While part-time attendance does not appear to be a major concern there, the teacher-child ratio decreases with the child's age. Thus, younger children affect the capacities of day care centers more than older ones and can be thought of as having a larger size.

and submit preferences over pairs of hospitals (Roth, 1984). A couple requires two positions; however they may not be at the same hospital. Refugee resettlement (Delacrétaz, Kominers, and Teytelboym, 2016) constitutes another extension. People accepted as refugees in a given country are typically resettled across various local areas that provide them with multiple services (e.g., housing, school places, or training programs). Multidimensional constraints arise in this problem as families have different service requirements, depending for example on whether they have children or specific needs.

The aim of this paper is to pin down the implications of introducing sizes and develop suitable solution concepts. I study the simplest matching model with sizes. Agents (e.g., parents, students) have ordinal preferences over objects (e.g., day care centers, host universities) that are available in multiple identical units (e.g., part-time places in a specific day care center, semesters on exchange at a specific partner university). For each object, agents are ranked according to exogenous priorities. Agents can have a size of either one or two: *single-unit agents* require one unit of an object (e.g., families placing their child part-time, students going on exchange for a semester) and *double-unit agents* require two units of the same object (e.g., families placing their child full-time or students going on exchange for a year).

The model – as well as the insights gathered and concepts developed throughout the paper – can be extended in various ways to fit specific applications. Extensions can follow at least three directions. First, constraints may take place over several dimensions. Children going to day care part-time typically attend specific days, for example Monday-Wednesday-Friday or Tuesday-Thursday. Refugee resettlement involves such multidimensional constraints, each of which is a service that local areas can provide to refugees. Second, the model can be extended to the case where agents have preferences over both an object and a number of units. Students may prefer to go on exchange for a year but, if this is not possible, be willing to go for only a semester. Third, agents may desire units of different objects. Doctors in a couple can work in two different hospitals located in the same city. Sizes take different forms in each of these applications; however they are present in all of them. Designing central clearing houses for these markets requires an understanding of how sizes affect matching problems. This paper provides important insights on the subject and serves as a stepping stone towards developing new solutions for a wide-range of matching applications.

Stability – initially introduced by Gale and Shapley (1962) – is a central concept in matching theory. In this paper's model, an agent and an object form a *blocking pair* of a given matching if the agent prefers the object to his own and at least the number of units he requires are either unassigned or assigned to agents with a lower priority. A matching is stable if it does not have any blocking pair. Stable matchings are desirable for a range of reasons that vary depending on the application. In two-sided matching markets, with strategic agents on both sides, stability constitutes an essential equilibrium criterion. For instance, in the NRMP, a doctor and a hospital that form a blocking pair have an incentive to match with one another outside of the matching program; thus an unstable matching is not at equilibrium. In matching markets with priorities such as school choice, day care, student exchange, or refugee resettlement, it constitutes a fairness criterion.[4] A blocking pair is unfair in the sense that the agent is not able to get an object even though he has a high enough priority. Throughout the paper, I treat stability as a fairness criterion; however, the solution concepts I develop may prove useful to two-sided matching markets as well.

In a matching market where all agents have the same size (e.g., school choice), the set of stable matchings contains an *agent-optimal stable matching* (Gale and Shapley, 1962), that is a stable matching that weakly dominates all other stable matchings. This is no longer true in the presence of sizes. The set of stable matchings may be empty (Example 1) and, if nonempty, it may contain multiple *agent-undominated stable matchings* – that is, a stable matching not dominated by any other – instead of an agent-optimal stable matching (Example 2).

I characterize stability using three axioms, each of which constitutes a fairness criterion. A matching is **size envy-free** if, whenever an agent prefers an object to his own, all agents matched to that object have either a higher priority or a smaller size. Size envy-freeness is a fairness criterion in the sense that any priority violation can be justified by the agents' different sizes. For any nonnegative integer $K$, a matching is **$K$-bounded** if, whenever an agent prefers an object to his own, at most $K$ units of the object are either unassigned or assigned to agents with a lower priority. $K$-boundedness constitutes a fairness criterion in the sense that it bounds the number of units that an agent can claim given his priority. Of particular interest throughout is 1-*boundedness*, the special case where $K = 1$ and agents can claim at most one unit per object. A matching is **non-wasteful** if, whenever an agent prefers an object to his own, the object has enough unassigned units for the agent to be matched to it without removing any other agent. Non-wastefulness constitutes both a fairness and an efficiency criterion as it ensures that units only remain unassigned if they cannot benefit any agent. I show that a matching is stable if and only if it is size envy-free, 1-bounded, and non-wasteful.

I show (Theorem 1) that two of these axioms are incompatible with one another: for any $K$, it is possible to construct a market in which there does not exist any $K$-bounded and non-wasteful matching. In contrast, there always exists a 1-bounded and size envy-free matching

[4]Abdulkadiroglu and Sönmez (2003) refer to stability as the *elimination of justified envy* to highlight this fairness interpretation.

as well as a non-wasteful and size envy-free matching. Guided by these results, I develop two fairness criteria. **Unit-stability** allows those blocking pairs where a single-unit agent prefers an object to his own and exactly one unit of that object is unassigned but precludes all other blocking pairs. It is characterized by 1-boundedness and size envy-freeness, as such it relaxes stability but guarantees existence and provides a clear fairness criterion. **Size-stability** allows those blocking pairs where a double-unit agent prefers an object to his own and single-unit agents with a lower priority are matched to that object. It is characterized by non-wastefulness and size envy-freeness, and therefore it also guarantees existence and provides a clear fairness criterion, though a less stringent one than stability.

These results point to an important trade-off that market designers face in matching markets with sizes. It is possible to either bound the instability to a limited number of units per object – in fact, it is possible to limit it to one unit per object – or to eliminate waste, but not both. Unit-stability achieves the former and size-stability the latter. Which of the two fairness criteria is more desirable depends on each application and the relative importance of respecting priorities versus eliminating waste. For example, Delacrétaz, Kominers, and Teytelboym (2016) argue that waste can be tolerated in refugee resettlement while respecting priority is key and develop a solution concept that, in this paper's model, is unit-stable (though their fairness criterion is slightly less permissive). On the other hand, universities may prefer to send lower-priority students on exchange for a semester rather than to leave some positions unused.

In a matching market with priorities, a natural solution concept is to maximize welfare subject to a suitable fairness criterion. **Undominated unit-stable matchings** (UUSMs), that is unit-stable matchings that are not dominated by any other unit-stable matchings, achieve this goal and, therefore, constitute a natural solution concept for applications where waste can be tolerated. **Undominated size-stable matchings** (USSMs) – size-stable matchings that are not dominated by any other size-stable matchings –, in contrast, may be unfair to double-unit agents because there does not exist any bound on how many single-unit agents may violate their priority. I propose to refine the solution concept by only considering those USSMs that are **d-undominated**, that is undominated not only in general but also from the point of view of double-unit agents. While the priority of double-unit agents may still be violated, they are compensated by the fact that the matching is as good as possible for them: any size-stable matching that makes a double-unit agent better-off makes another one worse-off.

Relaxing stability allows recovering existence but also matters from a welfare point of view. Any fairness criterion limits the set of matchings that may be selected and, as such, may have consequences on welfare. Thus, a fairness criterion that is more permissive than stability can allow recovering higher-welfare matchings. I formalize this by introducing **set domination** to compare sets of matchings. Set of matching $M_1$ weakly set dominates set of matchings $M_2$ if every matching in $M_2$ is weakly dominated by at least one matching in $M_1$ and no matching in $M_1$ is strictly dominated by any matching in $M_2$.[5] The sets of undominated unit-stable and size-stable matchings both dominate the set of undominated stable matchings, a consequence of the fact that both fairness criteria relax stability. Perhaps surprisingly, I show that the set of undominated size-stable matchings set dominates the set of undominated unit-stable matchings (Theorem 2). This result adds an additional consideration to the trade-off between bounding instability and eliminating waste: the latter is more costly from a welfare point of view.

**Related Literature**

An important body of literature provides "nearly" stable solutions in matching markets with sizes. Biró and McDermid (2014) study an extension of the present model where agents' sizes lie between 1 and $n \geq 2$. They show that, if the quota of each object is increased or decreased by at most $n-1$ units, then an algorithm exists that finds a stable matching.[6] Dean, Goemans, and Immorlica (2006) and Yenmez (2018) each propose an algorithm based on deferred acceptance that are achieves these bounds. The matching produced by Dean, Goemans, and Immorlica's (2006) algorithm is stable if the capacity of each object is increased by at most $n-1$ units[7] and the one produced by Yenmez' (2014) algorithm is stable if the capacity of each object is decreased by at most $n-1$ units.[8] In the present model, $n = 2$ so a stable matching can be obtained by adding or removing one unit. Nguyen and Vohra (2018) derive a similar result for the couples problem. Increasing or decreasing the quota of each object (hospital) by at most two units allows finding a market where a stable matching exists. In addition, the sum of quotas does not decrease and increases by at most four units. In this paper, I assume strict quotas and do not consider solutions that require increasing them. However, as I show in the appendix (Proposition 7), the existence of unit-stable matchings implies that of matchings which can be made stable by discarding at most one unit per object

[5] Strict set domination obtains if, in addition, the two sets are distinct.

[6] In the context of minimum quotas, Fragiadakis and Troyan (2017) propose an algorithm that finds a stable matching by removing some units of capacity.

[7] Cseh and Dean (2016) adapt that algorithm in order to find a matching that minimizes the total number of units that need to be added.

[8] Yenmez (2018) studies a model of college admissions with contracts. Sizes are introduced by the fact that students can take up either a full place or a fraction $1/n$ of a place. Stability obtains by assuming that colleges do not need to fill up their capacity, instead they reject students as soon as they have less than a full place available. This can be translated into the model of Dean, Goemans, and Immorlica (2006) and Biró and McDermid (2014) by letting a full place be equal to $n$ units of capacity. Then, a college stops filling up its quota once it has $n-1$ or less units available.

but the converse does not hold.

As mentioned above, the National Resident Matching Program and refugee resettlement are two real-world matching markets with sizes that have been studied and each of them constitutes a different extension of the model studied in this paper. As a result, the trade-off between bounding instability and eliminating waste is relevant for these two markets. An "engineering" solution was successfully implemented for the NRMP (Roth and Peranson, 1997, 1999; Roth, 2003) and, though existence is not guaranteed, it has found a stable matching every year since its inception in 1983.[9] This may of course not be the case for other matching markets with sizes and the focus of the present paper is on existence in general. Delacrétaz, Kominers, and Teytelboym (2016) propose *envy-freeness* as a fairness criterion and show the existence of an agent-optimal envy-free matching in their model (of which the model studied in this paper is a special case). Kamada and Kojima (2018) provide a sufficient and almost necessary condition on capacity constraints for the existence of an agent-optimal envy-free matching, which is satisfied by the setup of Delacrétaz, Kominers, and Teytelboym (2016) and, consequently, this paper.[10] A matching is envy-free if, whenever an agent prefers an object to his own, all agents matched to that object have a higher priority. In the absence of sizes, the agent-optimal envy-free matching is the agent-optimal stable matching. In the model studied in this paper, the agent-optimal envy-free matching is unit-stable but not necessarily undominated. The reason is that a double-unit agent can envy a single-unit agent in a unit-stable matching, hence unit-stable matchings are not necessarily envy-free. Stable matchings are not necessarily envy-free either for the same reason.

Last, several papers have studied similar applications to those mentioned in this paper but do not consider agents of different sizes. Kennes, Monte, and Tumennasan (2014) study the assignment of children to day care centers in a dynamic context, taking into account the fact that children may move from one center to another once they have secured a place. The authors do not consider the part-time feature of day care and effectively build a dynamic extension of the school choice model, framed in the context of day care. Considering the dynamic aspect of day care is certainly worthwhile; however the success encountered by the reforms of school choice systems across the world suggests that taking care of that static problem can already greatly improve the way the market operates. A dynamic model that caters for agents of different sizes could then lead to further improvements. Dur and Ünver

[9]Potential explanations for the existence of a stable matching in practice include the market's large size and relatively small proportion of couples, the fact that each doctor lists only a small fraction of the hospitals (and vice-versa), and the fact that preferences on both sides are correlated. These conditions may of course vary from one application to another.

[10]Delacrétaz, Kominers, and Teytelboym (2016) and Kamada and Kojima (2018) respectively use the terms *quasi-stable* and *fair* for what I call envy-free, following Roth and Wu (2018).

(2019) consider a different aspect of student exchange.[11] Their emphasis lies on the balance of students between the two partners and its impact on exchange agreements in a dynamic environment. Instead, I take the terms of the agreements as given and focus on the university's decision about which students it will send to its partners. Sönmez (2013) and Sönmez and Switzer (2013) study a matching market between cadets and branches in the American Army. A length of time is specified for every matched pair, which is somewhat reminiscing of children attending day care part-time or full time, or of students going on exchange for a semester or a year. The two setups, however, do not resemble one another beyond the fact that they model a matching market. The duration of a contract between a cadet and a branch is designed to give cadets an incentive to commit for a longer period and does not impact the number of cadets with which each branch may be matched. Budish and Cantillon (2012) and Kojima (2013) study the matching of college students to classes. Students attend multiple (typically four) classes but student preferences over classes are assumed to be responsive, which ensures the existence of a student-optimal stable matching.

The remainder of the paper is organized as follows. Section 2 formally presents the model. Section 3 defines stability and characterizes it as the combination of three axioms, two of which are incompatible with one another. Section 4 introduces unit- and size-stability as fairness criteria and characterizes each of them as the combination of two of the three axioms that make up stability. Section 5 compares the proposed solution concepts in terms of welfare and Section 6 concludes. The appendix contains all proofs and some additional results.

# 2 Model

There are a set $A$ of **agents** and a set $O$ of **objects**. Each object $o \in O$ is available in $q_o \geq 1$ identical and indivisible units. I refer to $q_o$ as the **quota** of object $o$ and define the **quota vector q** to be the $|O|$-dimensional vector containing all quotas. The set of agents is partitioned into two subsets $S$ and $D$. Agents in $S$ are the **single-unit agents** and require one unit. Agents in $D$ are the **double-unit agents** and require two units of the same object.[12] Define $w_a \in \{1, 2\}$ such that $w_a \equiv 1$ if $a \in S$ and $w_a \equiv 2$ if $a \in D$ to be the **size** of agent $a$. The **size vector w** is the $|A|$-dimensional vector containing the size of all

[11] Dur and Ünver (2019) call this *tuition exchange*.

[12] In school choice, agents are students, objects are schools, and units are seats in a school. All students are single-unit agents. In day care, agents are children (or their parents), objects are day care centers and units are part-time places in a center. Single-unit agents are children who require a part-time place and double-unit agents are children who require a full-time place. In student exchange, agents are students, objects are host universities, and units are places available at a given host university. Single-unit agents are students who want to go on exchange for a semester and double-unit agents are students who want to go on exchange for a year.

agents. I assume the existence of a **null object**, denoted $\emptyset$, with a large enough quota to accommodate all agents: $q_\emptyset = \sum_{a\in A} w_a = 2|D| + |S|$.[13]

Agents have strict, transitive, and ordinal **preference relations** over all objects. The preference relation of agent $a$ is denoted $\succ_a$ and $o \succ_a o'$ signifies that agent $a$ prefers object $o$ to object $o'$. The weak preference relation $o \succeq_a o'$ means that either $o \succ_a o'$ or $o = o'$. The **preference profile** is the $|A|$-tuple of preference relations $\boldsymbol{\succ} \equiv (\succ_a)_{a\in A}$ containing the preference relations of all agents. For every object, agents are ranked in order of priority. $\rhd_o$ represents the **priority relation** of object $o$ and consists of a strict and transitive ranking of all agents. $a \rhd_o a'$ signifies that agent $a$ has a higher priority than agent $a'$ for object $o$. The weak priority relation $a \trianglerighteq_o a'$ means that either $a \rhd_o a'$ or $a = a'$. The **priority profile** is the $|O|$-tuple $\boldsymbol{\rhd} \equiv (\rhd_o)_{o\in O}$ containing the priority relations of all objects.[14] A **market** is a tuple $\langle A, O, \boldsymbol{\succ}, \boldsymbol{\rhd}, \mathbf{w}, \mathbf{q}\rangle$.

**Definition 1.** *A* ***matching*** *is a correspondence* $\mu : A \cup O \to A \cup O$ *such that, for all* $(a, o) \in A \times O$*,*

*(i)* $\mu(a) \in O$*,*

*(ii)* $\mu(o) \subseteq A$*,*

*(iii)* $\mu(a) = o$ *if and only if* $a \in \mu(o)$*, and*

*(iv)* $\sum_{a'\in\mu(o)} w_{a'} \leq q_o$*.*

Let $M$ denote the set of all matchings. Notice that Definition 1, which is otherwise standard, includes *feasibility* (part (iv)). The only matchings considered are those that are feasible in the sense that every object has enough units available to assign to all agents matched to it.

# 3 Stability

## 3.1 Definition

Given a matching $\mu$, let $\widehat{\mu}_a(o) \equiv \{a' \in \mu(o) \mid a' \rhd_o a\}$ be the set of agents who are matched to object $o$ at $\mu$ and have a higher priority than agent $a$ for that object. Agent $a$ has a **claim**

[13]This is without loss of generality, the null object can simply represent the possibility for an agent to remain unmatched.

[14]In order to keep the model as simple as possible, it is assumed that even the agent with the lowest priority can get an object so long as other agents do not compete for it. This assumption is natural in a model where the focus lies on agent welfare and without loss of generality since an eligibility threshold under which an agent cannot be matched to an object can be introduced by ranking the objects after the null one in the agent's preference relation.

to $n$ units of object $o$ at matching $\mu$ if $o \succ_a \mu(a)$ and $\sum_{a' \in \widehat{\mu}_a(o)} w_{a'} \leq q_o - n$; that is, $a$ prefers $o$ to his match and at least $n$ units of $o$ are either unassigned or assigned to an agent with a lower priority.

**Definition 2.** *$(a, o) \in A \times O$ is a **blocking pair** of matching $\mu \in M$ if $a$ has a claim to $w_a$ units of $o$ at $\mu$.*

In words, a single-unit agent $s$ and an object $o$ form a blocking pair if $s$ prefers $o$ to his current match and at least one unit of $o$ is either unassigned or assigned to an agent with a lower priority. A double-unit agent $d$ and an object $o$ form a blocking pair if $d$ prefers $o$ to his current match and at least *two* units of $o$ are either unassigned or assigned to an agent with a lower priority.[15]

**Definition 3.** *A matching is **stable** if it does not have any blocking pairs.*

In the absence of double-unit agents, Definition 3 is equivalent to the standard definition of stability (see, e.g., Gale and Shapley (1962) or Abdulkadiroglu and Sönmez (2003)). Introducing double-unit agents requires an element of caution as a one-unit claim is not enough to conclude that the agent's priority is violated. Double-unit agents can only be assigned two units and, therefore, a blocking pair only forms when they are able to claim at least two units of an object. This definition is consistent with the way stability has been defined in related models (see, e.g., McDermid and Manlove (2010), Biró and McDermid (2014), or Roth and Peranson (1999)).

Of particular interest are those stable matchings that are *undominated* in terms of agent welfare. **Domination** is a partial order on the set of matchings $M$: for any two matchings $\mu, \nu \in M$, $\mu$ dominates $\nu$ if $\mu(a) \succeq_a \nu(a)$ for all $a \in A$ and $\mu(a) \succ_a \nu(a)$ for some $a \in A$. In words, $\mu$ dominates $\nu$ if it makes all agents weakly better-off and at least one agent strictly better-off.[16] I write $\mu \succ \nu$ if $\mu$ dominates $\nu$ and $\mu \succeq \nu$ if $\mu$ weakly dominates $\nu$, that is if $\mu \succ \nu$ or $\mu = \nu$. The set of **undominated stable matchings (USMs)**, denoted $M^{US}$, contains all stable matchings that are not dominated by any other stable matching. If that set contains exactly one element, I refer to it as the **optimal stable matching**.[17]

[15] These units may be both unassigned, both assigned to the same lower-priority agent, assigned to two different lower-priority agents, or one may be unassigned while the other is assigned to a lower-priority agent.

[16] This criterion is often referred to as *Pareto* domination.

[17] These are often referred to as the agent-undominated stable matchings and the agent-optimal stable matching. Since there is no risk of confusion, I drop the reference to agents throughout the paper.

## 3.2 Preliminary Results

A classical result in matching theory is that, absent sizes, the set of stable matchings is nonempty and contains an optimal stable matching (Gale and Shapley, 1962).[18] As I show next, these results do not hold when agents have different sizes. As was first recognized by McDermid and Manlove (2010), a stable matching may not exist in this setup. Example 1 provides a market without stable matching.

**Example 1** (No Stable Matchings)**.** Let $S = \{s_1, s_2\}$, $D = \{d_1\}$, $O = \{o_1, o_2, \emptyset\}$, and $(q_{o_1}, q_{o_2}, q_\emptyset) = (2, 1, 4)$. The preferences and priorities are given below.

| $s_1$ | $s_2$ | $d_1$ |
|---|---|---|
| $o_1$ | $o_2$ | $o_1$ |
| $o_2$ | $o_1$ | $\emptyset$ |
| $\emptyset$ | $\emptyset$ | $o_2$ |

| $o_1$ (2) | $o_2$ (1) | $\emptyset$ (4) |
|---|---|---|
| $s_2$ | $s_1$ | |
| $d_1$ | $s_2$ | $\vdots$ |
| $s_1$ | $d_1$ | |

I show that the market presented in Example 1 does not have any stable matchings. If $d_1$ is matched to $o_1$, then $s_1$ is not by feasibility. As $s_1$ has the highest priority for $o_2$, a blocking pair is formed unless he is matched to it. In that case, however, $s_2$ is unmatched and forms a blocking pair with $o_1$. If $d_1$ is not matched to $o_1$, $s_2$ is, as otherwise $(d_1, o_1)$ constitutes a blocking pair. Then, $s_2$ and $o_2$ form a blocking pair unless $s_1$ is matched to $o_2$. In this case, however, $(s_1, o_1)$ is a blocking pair.

Next, Example 2 shows that a market may have multiple undominated stable matchings. While it is a result one might expect, to the best of my knowledge, it has not been formally shown for the model studied in this paper.

**Example 2** (Multiple USMs)**.** Let $S = \{s_1, s_2\}$, $D = \{d_1, d_2\}$, $O = \{o_1, o_2, \emptyset\}$, and $(q_{o_1}, q_{o_2}, q_\emptyset) = (2, 2, 6)$. The preferences and priorities are given below.

| $s_1$ | $s_2$ | $d_1$ | $d_2$ |
|---|---|---|---|
| $o_1$ | $o_2$ | $o_1$ | $o_2$ |
| $o_2$ | $o_1$ | $\emptyset$ | $\emptyset$ |
| $\emptyset$ | $\emptyset$ | $o_2$ | $o_1$ |

| $o_1$ (2) | $o_2$ (2) | $\emptyset$ (6) |
|---|---|---|
| $s_2$ | $s_1$ | |
| $d_1$ | $d_2$ | $\vdots$ |
| $s_1$ | $s_2$ | |
| $d_2$ | $d_1$ | |

The market presented in Example 2 has two undominated stable matchings:

$$\mu \equiv \begin{pmatrix} s_1 & s_2 & d_1 & d_2 \\ o_2 & o_2 & o_1 & \emptyset \end{pmatrix} \qquad \text{and} \qquad \mu' \equiv \begin{pmatrix} s_1 & s_2 & d_1 & d_2 \\ o_1 & o_1 & \emptyset & o_2 \end{pmatrix}.$$

[18] In fact, the set forms a lattice (Roth and Sotomayor, 1990).

First, suppose the existence of a stable matching in which $d_1$ is matched to $o_1$. Feasibility directly implies that no one else is, so $s_1$ is not matched to his first preference. As he has the highest priority for $o_2$, his second preference, stability dictates that he be matched to it. By feasibility, this in turns implies that $d_2$ is matched to the null. Finally, $s_1$ and $o_2$ form a blocking pair unless they are matched together since, in that case, $s_1$ would be matched to a less-preferred object one unit of $o_2$ would be unassigned. It follows that $\mu$ is the only stable matching (if any) in which $d_1$ is matched to $o_1$. I now show that $\mu$ is indeed stable. $s_1$ is matched to his second preference but does not have a claim to his first one, $o_2$, as both units of that object are assigned to the higher-priority agent $d_1$. $d_1$ is also matched to his second preference; however he only has a 1-unit claim to his first preference, $o_1$, as one unit of that object is assigned to the higher-priority agent $s_2$. Consequently, $\mu$ is the unique stable matching in which $d_1$ is matched to $o_1$.

Second, suppose the existence of a stable matching in which $d_1$ is not matched to $o_1$. As he finds $\emptyset \succ_{d_1} o_2$, $d_1$ is matched to the null in any such matching. In addition, $(d_1, o_1)$ constitutes a blocking pair unless $s_2$ is matched to $o_1$. In turn, stability dictates that $d_2$ be matched to $o_2$ as, otherwise, $s_2$ and $o_2$ form a blocking pair. Finally, $s_1$ has a claim to his first preference, $o_1$, unless he is matched to it. Then, no matching other than $\mu'$ is stable and matches $d_1$ to an object other than $o_1$. An analogous reasoning to the one above shows that $\mu'$ is stable.

It follows that the market presented in Example 2 has exactly two stable matchings: $\mu$ and $\mu'$. As $\mu$ favors $s_1$ and $d_2$ while $\mu'$ favors $s_2$ and $d_1$, they are both undominated stable matchings. The following proposition summarizes these preliminary results.

**Proposition 1.** *The set of stable matchings may be empty or contain multiple undominated stable matchings.*

Since domination is a partial order on the set of (stable) matchings, there exists at least one undominated stable matching if and only if the set of stable matching is nonempty.

## 3.3 Characterization

I next introduce three axioms that characterize stability and are crucial to the rest of the analysis. Given a matching $\mu$, agent $a$ **envies** agent $a'$ at $\mu$ if $\mu(a') \succ_a \mu(a)$ and $a \rhd_o a'$. That is, $a$ envies $a'$ if he would prefer to be matched to $\mu(a')$ and has a higher priority for it. A matching is **envy-free** if no agent envies any other agent. Notice that stable matchings do not necessarily preclude envy because a double-unit agent can have a one-unit claim to an object matched to one single-unit agent with a lower priority. The two stable matchings of Example 2 are not envy-free since in each of them a double-unit agent envies a single-unit agent.[19] In order to account for this possibility, the first axiom adapts envy-freeness to an environment with sizes.

**Axiom 1.** *A matching $\mu$ is* ***size envy-free*** *if, whenever an agent $a$ envies another agent $a'$ at $\mu$, $w_a > w_{a'}$.*

In words, double-unit agents only envy single-unit agents and single-unit agents do not envy anyone. This constitutes a fairness criterion in the sense that if an agent's priority is violated, this can be justified by his larger size. The second axiom bounds the size of claims that may exist at a given matching.

**Axiom 2.** *For any nonnegative integer $K = 0, 1, 2, \ldots$, a matching $\mu$ is* ***$K$-bounded*** *if there does not exist any $K+1$-unit claim at $\mu$.*

If a matching is $K$-bounded, every agent has the assurance that at most $K$ units of an object he prefers to his own are either unassigned or assigned to agents with a lower priority. This constitutes a fairness criterion as a matching may be seen as "reasonably fair" if only a small proportion of the units available are not assigned to the highest-priority agents. The third and last axiom is a common property in matching theory.

**Axiom 3.** *A matching $\mu$ is* ***non-wasteful*** *if, for any agent-object pair $(a, o)$ such that $o \succ_a \mu(a)$, $\sum_{a' \in \mu(o)} w_{a'} > q_o - w_a$.*

Waste refers to a particular kind of blocking pair where the agent can be matched to the object without displacing any other agent; as such, non-wastefulness constitutes both a fairness and an efficiency criterion. The next result formalizes the relationship between stability and the above axioms.

**Proposition 2.** *A matching is stable if and only if it is size envy-free, 1-bounded, and non-wasteful.*

## 3.4 Existence Results

A direct consequence of the possible nonexistence of stable matchings (Proposition 1) and the above characterization (Proposition 2) is that the existence of a matching satisfying all

[19] This of course does not apply when all agents have the same size, in which case a matching is stable if and only if it is envy-free and non-wasteful (Axiom 3), see, e.g., Roth and Wu (2018). In a model with general capacity constraints (of which the model studied in this paper in a special case), Kamada and Kojima (2018) *define* stability to be the combination of envy-freeness and non-wastefulness. For the purpose of the current paper, this does not appear to be the natural definition as it would preclude the two stable matchings of Example 2. In fact, as noted in Section 3.1, the definition used here is consistent with the literature on matching markets with sizes and couples.

three axioms is not guaranteed. The next result shows that this incompatibility arises from one pair of axioms.

**Theorem 1.**

*(i) For every $K \geq 0$, the set of $K$-bounded, and non-wasteful matchings may be empty;*

*(ii) The set of size envy-free and 1-bounded matchings is nonempty;*

*(iii) The set of size envy-free and non-wasteful matchings is nonempty.*

Theorem 1 points to a key trade-off in matching markets with sizes. On the one hand, if waste is to be completely eliminated, then it is not possible to bound the size of claims: an agent may prefer an object to his own while an arbitrary number of units of that object are assigned to lower-priority agents. On the other hand, if waste can be tolerated, then claims can be bounded to one unit per object: if an agent prefers an object to his own, then all but one units of that object are assigned to higher-priority agents. In Section 4, I propose two solution concepts that lie on either side of this trade-off. Which is more desirable depends on the application at hands. Delacrétaz, Kominers, and Teytelboym (2016) argue that waste is tolerable in refugee resettlement as groups of refugees (agents) arrive regularly and any unused capacity can benefit the next group. Goodwill from local areas (objects) is on the other hand of the utmost importance as they are the one providing service capacities (quotas). As I show below, they propose a solution that, in the model studied in this paper, is size envy-free and 1-bounded. At the other end of the spectrum, student exchange is a "one-shot" market in the sense that the number of students that a university can send to its partners does not increase if some places were not used in previous years. A university's exchange office may then prefer to send a lower quality applicant rather than no one at all.[20] Eliminating waste may also constitute a key requirement in a market with agents on both sides, such as the National Resident Matching Program (NRMP). Participants may rapidly lose trust in the matching system if a hospital does not fill a position that eligible applicants would be willing to take. The problem may be less severe if the hospital has been able to fill the position with a lower-quality applicant. Day care lies somewhere in between. While children may join at any point, the largest intake takes place once a year when older children go to school. At this point, it may be tolerable to underuse the capacity of day care centers in the hope that these places be filled with children who apply later. The cost of doing so

[20] This may of course depend on a university's specific circumstances. Dur and Ünver (2019) argue that some agreements specify that each partner must send approximately the same number of students over a moving time window. Such a feature could motivate an exchange office to send fewer students one year in order to send more of them later on. In practice, this requirement appears in the United States but is uncommon outside. For example, it does not play a role in Erasmus agreements.

Algorithm 1: Priority-Focused Deferred Acceptance (PFDA)

Round $i \geq 1$:

Every agent $a$ proposes to his most preferred object that has not rejected him.

Object $o$ rejects the proposal of agent $a$ if $a$'s size added to the sizes of all agents who have proposed to $o$ in some Round $j \leq i$ exceeds $o$'s quota. Otherwise, $o$ tentatively accepts $a$'s proposal.

If at least one proposal has been rejected, continue to Round $i+1$. Otherwise, end and match each agent with the last object to which he proposed.

may however be large as demand typically greatly exceeds supply. Whether it outweighs the benefit of limiting the size of claims depends on specific circumstances that are beyond the scope of this paper. While the formal proof can be found in the appendix, the remainder of this section is devoted to providing intuition for Theorem 1.

Delacrétaz, Kominers, and Teytelboym (2016) and Kamada and Kojima (2018) propose *envy-freeness*[21] as a fairness criterion. Their results imply the existence of an optimal envy-free matching, which can be found using the Priority-Focused Deferred Acceptance (PFDA) algorithm from Delacrétaz, Kominers, and Teytelboym (2016). I present in Algorithm 1 a version of PFDA adapted to the model studied in this paper. The PFDA algorithm follows a similar structure to Gale and Shapley's Deferred Acceptance (DA) algorithm with the difference that an object rejects an agent if his size added to those of higher-priority agents who have already proposed to the object (in the current round or a previous one) exceeds the object's quota. As a result, an object may reject a single-unit agent even though one unit remains unassigned so the matching produced may waste up to one unit of each object; nevertheless, it satisfies the other two axioms.

**Proposition 3.** *The optimal envy-free matching is size envy-free and* 1*-bounded.*

Proposition 3 directly implies part (ii) of Theorem 1. Another simple algorithm produces a matching that satisfies the other compatible pair of axioms. The Two-Stage Deferred Acceptance (TSDA, Algorithm 2) first considers single-unit agents and calculates the optimal stable matching for them. (Since all agents have the same size that matching exists and can be

[21] As noted in the introduction (footnote 10), Delacrétaz, Kominers, and Teytelboym (2016) and Kamada and Kojima (2018) respectively use the terms *quasi-stable* and *fair*.

Algorithm 2: Two-Stage Deferred Acceptance (TSDA)

Stage 1:

Leave double-unit agents unmatched and calculate the optimal stable matching for single-unit agents.

Stage 2:

Remove single-unit agents and the units they are assigned and calculate the optimal stable matching for double-unit agents in that modified market.

calculated using Gale and Shapley's (1962) DA algorithm.) Second, the algorithm considers double-unit agents and the units that have not been assigned to a single-unit agent and calculates the optimal stable matching for that modified market. Since the optimal stable matching is calculated in each stage, no unit is wasted and agents of the same size do not envy one another. In addition, single-unit agents do not envy double-unit agents since the latter may only be assigned units that the former do not want. The next result follows naturally.

**Proposition 4.** *The TSDA algorithm produces a size envy-free and non-wasteful matching.*

Proposition 4 directly implies part (iii) of Theorem 1. Part (i) is shown by counterexample. In order to provide some intuition for this impossibility result, I present below an example that covers the case where $K = 1$. It can be extended to arbitrarily large $K$'s, see Example 5 in the appendix.

**Example 3** (No 1-bounded non-wasteful Matching)**.** Let $S = \{s_1, \widehat{s}_1, \widehat{s}_2\}$, $D = \{d_1\}$, $O = \{o_1, o_2, \emptyset\}$, and $(q_{o_1}, q_{o_2}, q_\emptyset) = (2, 2, 5)$. The preferences and priorities are given below.

| $s_1$ | $\widehat{s}_1$ | $\widehat{s}_2$ | $d_1$ |
|---|---|---|---|
| $o_1$ | $o_2$ | $o_2$ | $o_1$ |
| $o_2$ | $o_1$ | $o_1$ | $\emptyset$ |
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $o_2$ |

| $o_1$ (2) | $o_2$ (2) | $\emptyset$ (5) |
|---|---|---|
| $\widehat{s}_1$ | $s_1$ | |
| $\widehat{s}_2$ | $\widehat{s}_1$ | $\vdots$ |
| $d_1$ | $\widehat{s}_2$ | |
| $s_1$ | $d_1$ | |

It needs to be shown that every non-wasteful matching in Example 3 has a 2-unit claim. If $d_1$ is matched to $o_1$, then feasibility dictates that none of the single-unit agents be matched to $o_1$ and at least one of them be matched to $\emptyset$. If $s_1$ is matched to $\emptyset$, he has a 2-unit claim to $o_2$ as he has the highest priority for that object. If either $\widehat{s}_1$ or $\widehat{s}_2$ is matched to $\emptyset$, he has a 2-unit claim to $o_1$ as both units of that object are assigned to $d_1$, who has a lower priority. If $d_1$ is matched to $o_2$, he has a claim to at least two unassigned units of $\emptyset$, making the matching wasteful. It follows that $d_1$ is matched to $\emptyset$ in any 1-bounded and non-wasteful matching. In that case, $d_1$ has a 2-unit claim to $o_1$ unless at least one of $\widehat{s}_1$ or $\widehat{s}_2$ is matched to that object. That agent has a claim to an unassigned unit of $o_2$ unless the other two single-unit agents are matched to $o_2$. The matching obtained is wasteful since one unit of $o_1$ is unassigned and $s_1$ prefers $o_1$ to $o_2$. Consequently, there does not exist any 1-bounded and non-wasteful matching in Example 3.

# 4 Two Solution Concepts

In this section, I propose two relaxations of stability – *unit-stability* and *size-stability* – that lie on either side of the trade-off identified by Theorem 1. The motivation is two-fold. First, an alternative solution concept is required to cater for instances where the set of stable matchings is empty. I show that the set of unit-stable matchings and the set of size-stable matchings are always nonempty. Second, stability is a rather restrictive fairness criterion, which can have important consequences on welfare. For example, Abdulkadiroğlu, Pathak, and Roth (2009) study the New-York high-school match, which selects the student-optimal stable matching, and show that on average 4,000 students could be made better-off each year without making any student worse-off.[22] Therefore, a suitable but less stringent fairness criterion is valuable from a welfare points of view. I argue that both solution concepts provide a clear fairness criterion and show that they both lead to welfare gains over stability. Perhaps surprisingly, I show that size-stability, which eliminates waste (Axiom 3) has better welfare properties than unit-stability, which bounds instability in the sense of Axiom 2.

## 4.1 Unit-stability

**Definition 4.** *$(a, o) \in A \times O$ is a* ***unit blocking pair*** *of matching $\mu \in M$ if it is a blocking pair of $\mu$ and either (i) $a \in D$, (ii) $\sum_{a' \in \mu(o)} w_{a'} \neq q_o - 1$, or (iii) $\sum_{a' \in \widehat{\mu}_a(o)} w_{a'} \neq q_o - 1$.*

In words, a single-unit agent $s$ and an object $o$ form a unit blocking pair if $s$ prefers $o$ to his current match and either at least one unit of $o$ is assigned to an agent with a lower priority or at least *two* units of $o$ are an unassigned. The definition does not change for

[22] In the school choice context, despite the fact that stable matchings are guaranteed to exist, several relaxations of stability have been proposed in order to improve welfare, see, e.g., Alcalde and Romero-Medina (2015), Ehlers and Morrill (2018), and Troyan, Delacrétaz, and Kloosterman (2018).

double-unit agents. Definition 4 strengthens the definition of blocking pair by ruling out the case where a single-unit agent prefers an object to his own while exactly one unit of that object is unassigned and all others are assigned to higher-priority agents.

**Definition 5.** *A matching is* ***unit-stable*** *if it does not have any unit blocking pairs.*

Unit-stability relaxes stability by allowing one specific type of blocking pairs. Since it rules out all other blocking pairs, unit-stability remains a clear fairness criterion. In fact, it is characterized by two of the three axioms that make up stability.

**Proposition 5.** *A matching is unit-stable if and only if it is size envy-free and* 1*-bounded.*

Propositions 3 and 5 directly imply that the PFDA algorithm produces a unit-stable matching and, therefore, the set of such matchings is nonempty. In Example 1, which does not have any stable matching, there are three unit-stable matchings:

$$\mu \equiv \begin{pmatrix} s_1 & s_2 & d_1 \\ o_1 & o_1 & \emptyset \end{pmatrix}, \quad \mu' \equiv \begin{pmatrix} s_1 & s_2 & d_1 \\ o_2 & o_1 & \emptyset \end{pmatrix}, \quad \text{and} \quad \mu'' \equiv \begin{pmatrix} s_1 & s_2 & d_1 \\ \emptyset & o_1 & \emptyset \end{pmatrix}.$$

$\mu''$ has three blocking pairs: $(s_1, o_1)$, $(s_1, o_2)$, and $(s_2, o_2)$. One unit of the $o_1$ is unassigned and the other is assigned to $s_2$, who has a higher-priority than $s_1$; therefore, $(s_1, o_1)$ is not a unit blocking pair. (Formally, $\sum_{a\in\mu(o_1)} w_a = \sum_{a\in\widehat{\mu}_{s_1}(o_1)} w_a = w_{s_2} = 1 = q_{o_1} - 1$.) Neither $(s_1, o_2)$ nor $(s_2, o_2)$ is a unit blocking pair since the unique unit of $o_2$ is unassigned. (Formally, $\sum_{a\in\mu(o_2)} w_a = \sum_{a\in\widehat{\mu}_{s_1}(o_2)} w_a = \sum_{a\in\widehat{\mu}_{s_2}(o_2)} w_a = 0 = q_{o_2} - 1$.) Therefore, $\mu''$ is unit-stable. $\mu$ and $\mu'$ are also unit-stable as they each have one of these blocking pairs, which for the same reason is not a unit blocking pair.

To see that these are the only unit-stable matchings, consider an alternative matching $\nu$. If $\nu(s_2) = \emptyset$, $(s_2, o_1)$ is a unit blocking pair since $s_2$ has a claim to both units of $o_1$ ($\sum_{a\in\widehat{\mu}_{s_2}(o_1)} w_a = 0 \neq 1 = q_{o_1} - 1$). If $\nu(s_2) = o_2$, then $\nu(d_1) = o_1$ as otherwise $(d_1, o_1)$ is a blocking pair and $d_1 \in D$ implies it is a unit blocking pair. Then, $\nu(s_1) = \emptyset$ so $(s_1, o_2)$ is a blocking pair. Since the unique unit of $o_2$ is matched to lower-priority agent $s_2$ ($\sum_{a\in\mu(o_2)} w_a = w_{s_2} = 1 \neq 0 = q_{o_2} - 1$), it is a unit blocking pair and $\nu$ is not unit-stable. It follows that $\nu(s_2) = o_1$, which directly implies that $\nu(d_1) = \emptyset$. Depending on $\nu(s_1)$, this leaves three options: $\mu$, $\mu'$, and $\mu''$.

Notice that welfare differs across the three matchings. While $\mu(s_2) = \mu'(s_2) = \mu''(s_2) = o_1$ and $\mu(d_1) = \mu'(d_1) = \mu''(d_1) = \emptyset$, $\mu(s_1) = o_1 \succ_{s_1} \mu'(s_1) = o_2 \succ_{s_1} \mu''(s_1) = \emptyset$. Therefore, $\mu$ dominates $\mu'$ and $\mu'$ dominates $\mu''$. Given a fairness criterion, it is natural to maximize welfare under the constraint that the criterion is satisfied. Therefore, the natural solution concept here is an **undominated unit-stable matching (USSM)**, that is a unit-stable

| Round 1 | | Round 2 | | Round 3 | | Round 4 | |
|---|---|---|---|---|---|---|---|
| $s_1 \to o_1$ | ✗ | $s_1 \to o_2$ | ✓ | $s_1 \to o_2$ | ✓ | $s_1 \to o_2$ | ✓ |
| $s_2 \to o_2$ | ✓ | $s_2 \to o_2$ | ✗ | $s_2 \to o_1$ | ✓ | $s_2 \to o_1$ | ✓ |
| $d_1 \to o_1$ | ✓ | $d_1 \to o_1$ | ✓ | $d_1 \to o_1$ | ✗ | $d_1 \to \emptyset$ | ✓ |

Table 1: PFDA mechanism on Example 1.

matching that is not dominated by any other unit-stable matching. Since the set of unit-stable matchings is nonempty and domination is a partial order on the set of matchings, the set of undominated unit-stable matchings, denoted $M^{UUS}$, is nonempty. In the case of Example 1, $\mu$ is the only such matching. In other cases, there may be multiple ones, for instance one can verify that the two stable matchings of Example 2 are also the two undominated unit-stable matchings.

Table 1 displays the steps of the PFDA algorithm on Example 1 and shows that it produces the dominated unit-stable matching $\mu'$. The reason is that envy-freeness is a stronger requirement than size envy-freeness since it precludes situations where a double-unit agent envies a single-unit agent. As the PFDA algorithm produces an envy-free and 1-bounded matching, replacing envy-freeness by size envy-freeness may allow finding additional matchings, which may dominate it. This is the case of $\mu$, which is unit-stable but not envy-free: $s_1$ is matched to $o_1$ even though $d_1$ prefers $o_1$ to his object ($\emptyset$) and has a higher priority than $s_1$ for $o_1$. Both solution concepts limit claims to one unit per object at the price of potentially creating waste. Which one should be used (if any) depends on the application at hand. Envy-freeness strictly respects priority as it precludes any violation, even those that do not create a blocking pair (when a double-unit agent envies a single-unit agent) Delacrétaz, Kominers, and Teytelboym (2016) argue that this is very important in refugee resettlement and follow that approach. In other applications, welfare concerns may take precedence. For example, it may well be acceptable for a part-time child to attend a day care center even though a full-time child with a higher priority has not received a place.

## 4.2 Size-stability

**Definition 6.** *$(a, o) \in A \times O$ is a **size blocking pair** of matching $\mu \in M$ if it is a blocking pair of $\mu$ and either (i) $a \in S$ or (ii) $2 \cdot |\widehat{\mu}_a(o) \cap D| + |\mu(o) \cap S| \leq q_o - 2$.*

A single-unit agent and an object form a size blocking pair if they form a blocking pair. A double-unit agent $d$ and an object $o$ form a size blocking pair if $d$ prefers $o$ to his current match and two units of $o$ are either unassigned or assigned to *double-unit* agents with a lower priority. Definition 4 strengthens the definition of blocking pair by ruling out the case where a double-unit agent envies single-unit agents.

**Definition 7.** *A matching is* ***size-stable*** *if it does not have any size blocking pairs.*

Size-stability relaxes stability by disregarding claims of double-unit agents over units held by single-unit agents. Similarly to unit-stability, size-stability is characterized by two of the three axioms that make up stability.

**Proposition 6.** *A matching is size-stable if and only if it is size envy-free and non-wasteful.*

Propositions 4 and 6 directly imply that the TSDA algorithm produces a size-stable matching, hence the set of such matchings is nonempty. Since domination is a partial order on the set of matchings, this guarantees the existence of at least one **undominated size-stable matching (USSM)**, that is a size-stable matching that is not dominated by any other size-stable matching and, therefore, maximizes welfare subject to size-stability. In other words, the set of undominated size-stable matchings, denoted $M^{USS}$, is nonempty.[23]

In Example 1, the unique size-stable matching is

$$\mu^* \equiv \begin{pmatrix} s_1 & s_2 & d_1 \\ o_1 & o_2 & \emptyset \end{pmatrix}.$$

To see that $\mu^*$ is size-stable, notice that its only blocking pair is $(d_1, o_1)$; however, $(d_1, o_1)$ is not a size blocking pair because one of $o_1$'s two units is assigned to single-unit agent $s_1$. (Formally, $\widehat{\mu}^*_{d_1}(o_1) \cap D = \varnothing$ and $\mu^*(o_1) \cap S = \{s_1\}$ so $2 \cdot |\widehat{\mu}^*_a(o_1) \cap D| + |\mu^*(o_1) \cap S| = 1 > 0 = q_{o_1} - 2$.)

To see that $\mu^*$ is the unique size-stable matching, let $\nu$ be a size-stable matching. If $\nu(d_1) = o_1$, $(s_1, o_2)$ is a size blocking pair unless $\nu(s_1) = o_2$ but then $\nu(s_2) = \emptyset$ and $(s_2, o_1)$ is a size blocking pair. Therefore, any size-stable matching $\nu$ is such that $\nu(d_1) = \emptyset$. (As $\emptyset \succ_{d_1} o_2$, $(d_1, \emptyset)$ is a size blocking pair if $\nu(d_1) = o_2$.) In that case, $(s_1, o_1)$ is a size blocking pair unless $\nu(s_1) = o_1$ and, in turn, $(s_2, o_2)$ is a size blocking pair unless $\nu(s_2) = o_2$ so $\nu = \mu^*$.

Size-stability weakens stability but still constitutes a fairness criterion. Waste is eliminated and, while blocking pairs are not entirely precluded, they can only arise in a specific case: a double-unit agent prefers an object to his own and all but one unit of that objects are assigned to agents who have either a higher priority or a smaller size. A double-unit agent may envy agents who have a lower priority but these agents also have a smaller size, which can provide a justification for the priority violation.

In Example 1, $\mu^*$ is a credible solution: while $d_1$ has a claim to both units of $o_1$, one of the units is assigned to a smaller-size agent and $d_1$ cannot be matched to $o_1$ without creating

[23]Note that there may be multiple USSMs. For instance, both stable matchings of Example 2 are USSMs.

a claim for one of the single-unit agents. Nevertheless, this is not always the case and size-stability needs to be refined in order to provide a credible fairness criterion. I illustrate this with a very simple example.

**Example 4** (Inadequate USSM)**.** Let $S = \{s_1\}$, $D = \{d_1\}$, $O = \{o_1, \emptyset\}$, and $(q_{o_1}, q_\emptyset) = (2, 3)$. The preferences and priorities are given below.

| $s_1$ | $d_1$ |
|---|---|
| $o_1$ | $o_1$ |
| $\emptyset$ | $\emptyset$ |

| $o_1$ (2) | $\emptyset$ (3) |
|---|---|
| $d_1$ | $\vdots$ |
| $s_1$ | |

There are two size-stable matchings in Example 4:

$$\mu \equiv \begin{pmatrix} s_1 & d_1 \\ \emptyset & o_1 \end{pmatrix} \quad \text{and} \quad \mu' \equiv \begin{pmatrix} s_1 & d_1 \\ o_1 & \emptyset \end{pmatrix}.$$[24]

Neither matching dominates the other so both $\mu$ and $\mu'$ are undominated size-stable matchings; nevertheless, I argue that they are not equal when it comes to fairness. On the one hand, $\mu$ appears to be a natural solution since $o_1$ is matched to the higher-priority agent. On the other hand, $\mu'$ entirely disregards $d_1$'s priority. This points to a potential problem with size-stability as a fairness criterion: it does not protect the priorities of double-unit agents in any way against single-unit agents. In fact, the TSDA algorithm produces an undominated size-stable matching but completely disregards the priority of double-unit agents who are only able to get the units that remain unassigned at the end of the first stage. In some cases, such as Example 1, size-stability requires that some priorities be violated but in others, such as Example 4, double-unit agents may be unnecessarily harmed if a "bad" USSM is selected.

To remedy the situation, I propose to add to size-stability a requirement about double-unit agent welfare. **d-Domination** is a partial order on the set of matchings $M$:[25] for any two matchings $\mu, \nu \in M$, $\mu$ d-dominates $\nu$ if either $\mu$ dominates $\nu$ or $\mu(d) \succeq_d \nu(d)$ for all $d \in D$ and $\mu(d) \succ_d \nu(d)$ for some $d \in D$. I write $\mu \succ^D \nu$ if $\mu$ d-dominates $\nu$ and $\mu \succeq^D \nu$ if $\mu$ weakly d-dominates $\nu$, that is if $\mu \succ^D \nu$ or $\mu = \nu$. The set of **d-undominated size-stable matchings (d-USSMs)**, denoted $M^{dUSS}$, contains all size-stable matchings that are not d-dominated by any other size-stable matching.

Domination implies d-domination but the reverse is not true; therefore $M^{dUSS}$ is a subset of $M^{USS}$. The idea behind the concept is that a d-undominated size-stable matching is not

[24]The only other matching in this example matches both agents to the null. That matching is wasteful, hence it is not size-stable.

[25]I verify in the appendix (Proposition 8) that d-domination is indeed a partial order over $M$.

only undominated in general but also from the point of view of double-unit agents: no other size-stable matching makes a double-unit agent better-off without making another one worse-off. Adding this requirement precludes inadequate USSMs, such as $\mu'$ in Example 4. While double-unit agents may have their priority violated, they are compensated by the fact that their welfare is maximized under the constraint of size-stability.

# 5 Welfare Comparison

Undominated unit-stable matchings and d-undominated size-stable matchings offer two alternative solution concepts in matching markets with sizes. Both provide a clear fairness and efficiency criterion and are characterized by two of the three axioms that make up stability. Which one should be used depends on the application at hands. If waste can be tolerated but priorities are important, unit-stability is preferable as it limits claims to at most one unit per object. If, on the other hand, eliminating waste is more important than respecting priorities, size-stability is preferable: it ensures that no unit is wasted, at the price of not bounding the size of claims. In this section, I argue that welfare concerns may speak in favor of size-stable matchings.

**Set Domination** is a partial order over the set of all matching subsets $2^M$.[26] For any two subsets $M_1, M_2 \in 2^M$ with $M_1 \neq M_2$, $M_1$ set dominates $M_2$ if

(i) For all $\mu_2 \in M_2$, there exists $\mu_1 \in M_1$ such that $\mu_1 \succeq \mu_2$ and

(ii) For all $\mu_1 \in M_1$, there does not exist any $\mu_2 \in M_2$ such that $\mu_2 \succ \mu_1$.

In words, $M_1$ set dominates $M_2$ if all matchings in $M_2$ are weakly dominated by at least one matching in $M_1$ and no matching in $M_1$ is dominated by a matching in $M_2$. I write $M_1 \succ M_2$ if $M_1$ dominates $M_2$ and $M_1 \succeq M_2$ if $M_1$ weakly dominates $M_2$, that is if $M_1 \succ M_2$ or $M_1 = M_2$. As the lemma below formalizes, set domination collapses to domination when both sets contain exactly one element.

**Lemma 1.** *For any $\mu_1, \mu_2 \in M$, $\{\mu_1\} \succ \{\mu_2\}$ if and only if $\mu_1 \succ \mu_2$.*

The three solution concepts studied thus far can be compared in terms of set domination:

**Theorem 2.** $M^{USS} \succeq M^{UUS} \succeq M^{US}$.

Theorem 2 has important consequences for a market designer concerned with welfare. The fact that $M^{UUS}$ and $M^{USS}$ set dominate $M^{US}$ is not particularly surprising. As was

[26] I verify in the appendix (Proposition 8) that set domination is indeed a partial order over $2^M$.

shown in Sections 4.1 and 4.2, both unit- and size-stability constitute a relaxation of stability, which allows including higher-welfare matchings among the acceptable ones.[27] Nevertheless, it pins down an important point: using a less stringent fairness criterion matters not only in term of existence but also when it comes to welfare.[28]

The fact that $M^{USS}$ set dominates $M^{UUS}$ is not necessarily obvious at first sight. Recall that size-stability is characterized by size envy-freeness and non-wastefulness (Proposition 6) and denote by $M^{USE}$ the set of undominated size envy-free matchings, that is the set of matchings that are size envy-free and not dominated by any other size envy-free matching. The key step is to realize that an undominated size envy-free matching is non-wasteful as otherwise it is possible to match an agent to an object he prefers without affecting any other agent; the resulting matching is size envy-free and dominates the original one. Therefore, $M^{USS} = M^{USE}$. Since unit-stability is characterized by size envy-freeness and 1-boundedness (Proposition 5), all unit-stable matchings are size envy-free and $M^{USS} = M^{USE} \succeq M^{UUS}$. Undominated size-stable matchings have a welfare advantage over undominated unit-stable matchings because they effectively only require to satisfy one of the axioms that make up stability. This provides a new insight on the trade-off created by Theorem 1: from a welfare points of view, eliminating waste is less costly than bounding instability among size envy-free matchings.

Example 1 illustrates the welfare difference between size- and unit-stability. Recall that

$$\mu^* = \begin{pmatrix} s_1 & s_2 & d_1 \\ o_1 & o_2 & \emptyset \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} s_1 & s_2 & d_1 \\ o_1 & o_1 & \emptyset \end{pmatrix}$$

are respectively the unique (undominated) size-stable and the unique undominated unit-stable matchings. Clearly, $\mu^* \succ \mu$; the reason is that size-stability allows giving $d_1$ a 2-unit claim to $o_1$, which in turn allows matching $s_2$ to his first preferences $o_2$.

Theorem 2 offers an alternative criterion to select among undominated size-stable matchings. Rather than requiring that such a matching be d-dominated, one might instead require that it weakly dominate at least one undominated unit-stable matching. While double-unit agents may get their priority violated, this additional criterion ensure that they are at least as well-off as they are at some UUSM. Theorem 2 implies that this criterion does not eliminate all undominated size-stable matchings; however, it may eliminate some undesirable ones. In Example 4, this is the case of the "bad" USSM $\mu'$.

[27]Recall that the optimal envy-free matching, denoted $\mu^{OEF}$, is unit-stable, hence it does not dominate any undominated unit-stable matching but may be dominated by some of them so $M^{UUS} \succeq \{\mu^{OEF}\}$.

[28]Weak set domination does not preclude the case where the sets are identical. In fact, in Example 2, $M^{US} = M^{UUS} = M^{USS}$. For completeness, I show in the appendix (Proposition 9) that cases do exist where all undominated stable matchings are dominated by a unit-stable, respectively a size-stable, matching.

# 6 Conclusion

This paper studies a simple extension of the canonical school choice model where agents require either one or two units of an object. This seemingly small difference has important consequences, in particular stable matchings may not exist. I characterize stability to be the combination of three properties: size envy-freeness, $K$-boundedness, and non-wastefulness. I show that the last two axioms are incompatible, in the sense that the set of $K$-bounded and non-wasteful matchings may be empty. This result identifies an important trade-off that market designers face when confronted to matching markets with sizes: the size of claims can be bounded or waste can be eliminated but not both. I propose two solution concepts that lie on either side of that trade-off and guarantee existence. If waste is tolerable, unit-stability – which is characterized by size envy-freeness and 1-boundedness – limits claims to at most one unit per object but may be wasteful. Otherwise, size-stability – characterized by size envy-freeness and non-wastefulness – eliminates waste at the cost of no longer bounding the number of units that double-unit agents may claim. In order to alleviate this problem, I propose the solution concept of d-undominated size-stable matchings, which are undominated from the point of view of double-unit agents, thereby compensating these agents for the possible violation of their priority. From a welfare point of view, I show that, while both fairness criteria allow improving upon stability, size-stability is more desirable than unit-stability.

The present paper opens several avenues for future research. First, the model can be extended to fit several real-world matching problems. Agents may have multidimensional requirements, have preferences over both objects and a number of units, or desire units of different objects. In all of these extensions, the trade-off between eliminating waste and bounding the size of claims remains relevant. Unit-stability and size-stability can be easily extended so long as agents can be ranked by sizes, that is if there is no situation where one agent requires more units than another agent for one dimension but fewer for another dimension. Second, given the possible multiplicity of solutions, it may be possible to refine what constitutes a "good" undominated unit-stable matchings or d-undominated size-stable matching. Example 2, where two perfectly symmetric matchings are the only two USMs, UUSMs, and USSMs, suggests that there does not always exist one "best" matching; nevertheless, it may be possible to eliminate some of the less desirable outcomes. Finally, this paper has remained silent on mechanisms that could find desirable matchings. McDermid and Manlove (2010) have established that it is not possible in general to find a stable matching in polynomial time, but whether this extends to undominated unit-stable matchings and d-undominated size-stable matchings remains an open question. So do the incentive properties of such mechanisms.

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# Appendix

## Proofs

**Proof of Proposition 2: (If)** Let $\mu$ be a size envy-free, 1-bounded, and non-wasteful matching. By definition, a double-unit agent and an object form a blocking pair if the former has a 2-unit claim to the latter. As $\mu$ is 1-bounded, there does not exist any blocking pair at $\mu$ that involves a double-unit agent. Because $\mu$ is size envy-free, single-unit agents do not envy any agent; therefore they only have claims to unassigned units. However, because $\mu$ is non-wasteful, single-unit agents do not have any claim at all so $\mu$ is stable as it does not have any blocking pair. $\square$

**(Only If)** Let $\mu$ be a matching that is not 1-bounded. Then, there exists an agent $a$ who has a 2-unit claim to an object $o$. As $w_a \leq 2$, $a$ and $o$ form a blocking pair and $\mu$ is not stable. Next, let $\mu$ be a matching that is not size envy-free. Then, there exist two agents $a$ and $a'$ such that $a$ envies $a'$ and $w_a \leq w_{a'}$. By definition, the fact that $a$ envies $a'$ implies $\mu(a') \succ_a \mu(a)$ and $a \rhd_{\mu(a')} a'$; therefore $a$ has a $w_{a'}$-unit claim to $\mu(a')$. As $w_a \leq w_{a'}$, $a$ has a $w_a$-unit claim to $\mu(a')$, which means that $(a, \mu(a'))$ is a blocking pair, hence $\mu$ is not stable. Finally, let $\mu$ be a wasteful matching. Then, there exists an agent $a$ who has a claim to $w_a$ unassigned units of some object $o$. This directly implies that $a$ has a $w_a$-unit claim to $o$; as a result, $(a, o)$ constitutes a blocking pair and $\mu$ is not stable. $\blacksquare$

**Proof of Theorem 1:** Parts (ii) and (iii) are directly implied by, respectively, Propositions 3 and 4. I prove part (i) by counterexample. Let $K = 1, 3, 5, \ldots$ be an odd positive integer, I show that the example below does not have any $K$-bounded and non-wasteful matching. (Note that setting $K$ to be odd is without loss of generality as it can be arbitrarily large and the fact that no $K$-bounded and non-wasteful matching exists implies that no $K-1$-bounded and non-wasteful matching exists.)

**Example 5** (No $K$-bounded non-wasteful Matching)**.** Let $S = \{s_1, \ldots, s_K, \widehat{s}_1, \ldots \widehat{s}_{2K}\}$, $D = \{d_1, \ldots, d_{(K+1)/2}\}$, $O = \{o_1, o_2, \emptyset\}$, and $(q_{o_1},\ q_{o_2},\ q_\emptyset) = (2K, 2K, 4K+1)$. The preferences and priorities are given below.

| $s_1, \dots, s_K$ | $\widehat{s}_1, \dots, \widehat{s}_{2K}$ | $d_1, \dots, d_{(K+1)/2}$ |
|---|---|---|
| $o_1$ | $o_2$ | $o_1$ |
| $o_2$ | $o_1$ | $\emptyset$ |
| $\emptyset$ | $\emptyset$ | $o_2$ |

| $o_1$ $(2K)$ | $o_2$ $(2K)$ | $\emptyset$ $(4K+1)$ |
|---|---|---|
| $\widehat{s}_1$ | $s_1$ | |
| $\vdots$ | $\vdots$ | |
| $\widehat{s}_{2K}$ | $s_K$ | |
| $d_1$ | $\widehat{s}_1$ | |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $d_{(K+1)/2}$ | $\widehat{s}_{2K}$ | |
| $s_1$ | $d_1$ | |
| $\vdots$ | $\vdots$ | |
| $s_K$ | $d_{(K+1)/2}$ | |

Towards a contradiction, suppose the existence of a $K$-bounded and non-wasteful matching $\mu$. First, consider the case where all double-unit agents are matched to $o_1$: $\mu(d_k) = o_1$ for all $k = 1, \dots, (K+1)/2$. Fix $n = 0, 1, \dots, K-1$ and let $\widehat{s}^1, \dots, \widehat{s}^n$ be $n$ distinct elements of $\{\widehat{s}_1, \dots, \widehat{s}_{2K}\}$ and $s^1, \dots, s^n$ be $n$ distinct elements of $\{s_1, \dots, s_K\}$. Suppose that, for all $i = 1, \dots, n$, $\mu(\widehat{s}^i) = o_1$ and $\mu(s^i) = o_2$. In words, at least $n$ agents in $\{\widehat{s}_1, \dots, \widehat{s}_{2K}\}$ are matched to $o_1$ (their second preference) and at least $n$ agents in $\{s_1, \dots, s_K\}$ are matched to $o_2$ (their second preference). Then, $K+1$ units of $o_1$ are assigned to double-unit agents and $n$ units of $o_1$ are assigned to $\widehat{s}^1, \dots, \widehat{s}^n$ so at most $2K - (K+1) - n = K - (n+1)$ units of $o_1$ may be assigned to the $K - n$ agents in $\{s_1, \dots, s_K\} \setminus \{s^1, \dots, s^n\}$. Consequently, there exists an agent $s^{n+1} \in \{s_1, \dots, s_K\} \setminus \{s^1, \dots, s^n\}$ who is not matched to $o_1$. At most $K-1$ units of $o_2$ may be assigned to agents who have a higher priority than $s^{n+1}$ for $o_2$; therefore, if $s^{n+1}$ is matched to the null object, he has a claim to at least $K+1$ units of $o_2$, contradicting the assumption that $\mu$ is $K$-bounded. As a result, $s^{n+1}$ is matched to $o_2$: $\mu(s^{n+1}) = o_2$. In turn, this means that at least $n+1$ units of $o_2$ are assigned to $s^1, \dots, s^n, s^{n+1}$ so at most $2K - (n+1)$ units may be assigned to the $2K - n$ agents in $\{\widehat{s}_1, \dots, \widehat{s}_{2K}\} \setminus \{\widehat{s}^1, \dots, \widehat{s}^n\}$. Consequently, there exists an agent $\widehat{s}^{n+1} \in \{\widehat{s}_1, \dots, \widehat{s}_{2K}\} \setminus \{\widehat{s}^1, \dots, \widehat{s}^n\}$ who is not matched to $o_2$. If $\widehat{s}^{n+1}$ is matched to the null object, he has a claim to the $K+1$ units of $o_1$ assigned to the double-unit agents and $\mu$ is not $K$-bounded. $\widehat{s}^{n+1}$ is therefore matched to $o_1$: $\mu(\widehat{s}^{n+1}) = o_1$. Then, at least $n+1$ agents in $\{\widehat{s}_1, \dots, \widehat{s}_{2K}\}$ are matched to $o_1$ and at least $n+1$ agents in $\{s_1, \dots, s_K\}$ are matched to $o_2$. As the above reasoning holds for every $n = 0, \dots, K-1$, it follows by induction that at least $K$ agents in $\{\widehat{s}_1, \dots, \widehat{s}_{2K}\}$ are matched to $o_1$. Then, $K$ units of $o_1$ are assigned to these agents and $K+1$ units are assigned to the double-unit agents, a contradiction since $o_1$'s quota is $2K$.

It remains to show that a contradiction also arises in the case where, for some $k =$

$1, \ldots, (K+1)/2$, $d_k$ is not matched to $o_1$. If $d_k$ is matched to $o_2$, he has a claim to at least two unassigned units of the null object, contradicting the assumption that $\mu$ is non-wasteful. Therefore, $d_k$ is matched to the null object: $\mu(d_k) = \emptyset$. If none of the agents in $\{\widehat{s}_1, \ldots, \widehat{s}_{2K}\}$ are matched to $o_1$, then only double-unit agents may be matched to $o_1$ and have a higher priority than $d_k$. As there are $(K-1)/2$ double-unit agents other than $d_k$, at most $K-1$ units of $o_1$ are assigned to agents with a higher priority than $d_k$. Consequently, $d_k$ has a claim to at least $K+1$ units of $o_1$, contradicting the assumption than $\mu$ is $K$-bounded. Therefore, there exists an agent $\widehat{s}^0 \in \{\widehat{s}_1, \ldots, \widehat{s}_{2K}\}$ who is matched to $o_1$: $\mu(\widehat{s}^0) = o_1$. Fix $n = 0, 1, \ldots, K-1$ and let $\widehat{s}^1, \ldots, \widehat{s}^n$ be $n$ distinct elements of $\{\widehat{s}_1, \ldots, \widehat{s}_{2K}\} \setminus \{\widehat{s}^0\}$ and $s^1, \ldots, s^n$ be $n$ distinct elements of $\{s_1, \ldots, s_K\}$. Suppose that, for all $i = 1, \ldots, n$, $\mu(\widehat{s}^i) = o_1$ and $\mu(s^i) = o_2$. Then, at least $n+1$ agents in $\{\widehat{s}_1, \ldots, \widehat{s}_{2K}\}$ are matched to $o_1$ so at most $2K - (n+1)$ of them are matched to $o_2$. Additionally, no double-unit agent is matched to $o_2$ as such an agent would have a claim to at least two unassigned units of the null object, contradicting the assumption that $\mu$ is non-wasteful. If exactly $n$ agents in $\{s_1, \ldots, s_K\}$ are matched to $o_2$, then at most $2K - (n+1) + n = 2K - 1$ units of $o_2$ are assigned altogether. It directly follows that at least one unit of $o_2$ is unassigned. As $\widehat{s}^0, \widehat{s}^1, \ldots, \widehat{s}^n$ are matched to $o_1$ but prefer $o_2$, $\mu$ is wasteful, a contradiction. There consequently exists an agent $s^{n+1} \in \{s_1, \ldots, s_K\} \setminus \{s^1, \ldots, s^n\}$ who is matched to $o_2$: $\mu(s^{n+1}) = o_2$. This means that at least $n+1$ agents in $\{s_1, \ldots, s_K\}$ are matched to $o_2$, hence at most $K - (n+1)$ of them are matched to $o_1$. Additionally, at most $(K-1)/2$ double-unit agents are matched to $o_1$ since $d_k$ is not; therefore at most $K-1$ units of $o_1$ are assigned to them. If exactly $n+1$ agents in $\{\widehat{s}_1, \ldots, \widehat{s}_{2K}\}$ are matched to $o_1$, then at most $K - (n+1) + K - 1 + n + 1 = 2K - 1$ units of $o_1$ are assigned altogether. This directly implies that at least one unit of $o_1$ is unassigned. As $s^1, \ldots, s^n, s^{n+1}$ are matched to $o_2$ but prefer $o_1$, $\mu$ is wasteful, a contradiction. There consequently exists an agent $\widehat{s}^{n+1} \in \{\widehat{s}_1, \ldots, \widehat{s}_{2K}\} \setminus \{\widehat{s}^0, \widehat{s}^1, \ldots, \widehat{s}^n\}$ who is matched to $o_1$: $\mu(\widehat{s}^{n+1}) = o_1$. Therefore, at least $n+2$ agents in $\{\widehat{s}_1, \ldots, \widehat{s}_{2K}\}$ are matched to $o_1$ and at least $n+1$ agents in $\{s_1, \ldots, s_K\}$ are matched to $o_2$. As the above reasoning holds for every $n = 0, \ldots, K-1$, it follows by induction that at least $K+1$ agents in $\{\widehat{s}_1, \ldots, \widehat{s}_{2K}\}$ are matched to $o_1$, which directly implies that at most $K-1$ of them are matched to $o_2$. As none of the double-unit agents are matched to $o_2$, it follows that at most $2K-1$ units of $o_2$ are assigned altogether, hence at least one is unassigned. Those of $\widehat{s}_1, \ldots, \widehat{s}_{2K}$ who are matched to $o_1$ have a claim to that unit, hence $\mu$ is wasteful, a contradiction. ■

**Proof of Proposition 3:** By definition, envy-freeness implies size envy-freeness and by Delacrétaz, Kominers, and Teytelboym (2016), the PFDA algorithm produces the optimal envy-free matching; therefore, it remains to show that the PFDA algorithm produces a 1-

bounded matching.

I begin by introducing three elements of notation that are useful throughout the proof. First, let $N$ be the total number of rounds of the PFDA algorithm. (Since at least one rejection occurs in each round but the last one and the number of agents and objects is finite, $N$ is finite.) Second, for any $a \in A$, any $o \in O$, and any $i = 1, \ldots, N$, let $\widehat{A}^i_{(a,o)}$ be the set of agents who have a higher priority than $a$ for $o$ and propose to $o$ in some Round $j \leq i$ of the PFDA algorithm. Third, again for any $a \in A$, any $o \in O$, and any $i = 1, \ldots, N$, let $R^i_{(a,o)} \equiv w_a + \sum_{\widetilde{a} \in \widehat{A}^i_{(a,o)}} w_{\widetilde{a}}$ be the total size of $a$ and the agents with a higher priority who have proposed to $o$ by Round $i$. By construction, $a' \trianglerighteq_o a$ implies $\widehat{A}^i_{(a',o)} \subseteq \widehat{A}^i_{(a,o)}$ and $R^i_{(a',o)} \leq R^i_{(a,o)}$ while $i' \leq i$ implies $\widehat{A}^{i'}_{(a,o)} \subseteq \widehat{A}^i_{(a,o)}$ and $R^{i'}_{(a,o)} \leq R^i_{(a,o)}$. In addition, by construction, $o$ rejects a proposal by $a$ in Round $i$ if and only if $R^i_{(a,o)} > q_o$.

Denote by $\mu$ the matching produced by the PFDA algorithm and, towards a contradiction, suppose that an agent $a$ has a 2-unit claim to an object $o$ at $\mu$, that is $o \succ_a \mu(a)$ and $\sum_{\widetilde{a} \in \widehat{\mu}_a(o)} w_{\widetilde{a}} \leq q_o - 2$. As $\mu$ is envy-free, $\widehat{\mu}_a(o) = \mu(o)$ so $\sum_{\widetilde{a} \in \mu(o)} w_{\widetilde{a}} \leq q_o - 2$. Without loss of generality, let $a$ be the agent with the highest priority for $o$ among those who prefer $o$ to their object. Then, $\widehat{A}^N_{(a,o)} = \mu(o)$ so

$$R^N_{(a,o)} = w_a + \sum_{\widetilde{a} \in \widehat{A}^N_{(a,o)}} w_{\widetilde{a}} = w_a + \sum_{\widetilde{a} \in \mu(o)} w_{\widetilde{a}} \leq w_a + q_o - 2 \leq q_o.$$

It follows that $R^i_{(a,o)} \leq R^N_{(a,o)} \leq q_o$ for all $i = 1, \ldots, N$ so $o$ does not reject $a$, which contradicts $o \succ_a \mu(a)$. ■

**Proof of Proposition 4:**

Let $\mu$ be the matching produced by the TSDA algorithm. I show that $\mu$ is size envy-free and non-wasteful.

**(Size Envy-free)** If a single-unit agent $s$ envies an agent $a$ at $\mu$, then $(s, \mu(a))$ is a blocking pair of the first-stage matching, a contradiction since that matching is stable. Similarly, if a double unit agent $d$ envies another double-unit agent $d'$, $(d, \mu(d))$ is a blocking pair of the second stage matching. □

**(Non-Wasteful)** If there exists a single-unit agent $s$ and an object $o$ such that $o \succ_s \mu(s)$ and $\sum_{a \in \mu(o)} w_a \leq q_o - 1$, then $(s, o)$ is a blocking pair of the first-stage matching. Similarly, if there exists a double-unit agent $d$ and an object $o$ such that $o \succ_d \mu(d)$ and $\sum_{a \in \mu(o)} w_a \leq q_o - 2$, then $(d, o)$ is a blocking pair of the second-stage matching. ■

**Proof of Lemma 1:** If $\{\mu_1\} \succ \{\mu_2\}$, then by definition $\mu_1 \succeq \mu_2$ and, as $\{\mu_1\} \neq \{\mu_2\}$,

$\mu_1 \neq \mu_2$ so $\mu_1 \succ \mu_2$. If $\mu_1 \succ \mu_2$, then $\{\mu_1\} \neq \{\mu_1\}$, every matching in $\{\mu_2\}$ is weakly dominated by a matching in $\{\mu_1\}$, and no matching in $\{\mu_2\}$ dominates any matching in $\{\mu_1\}$ so $\{\mu_1\} \succ \{\mu_2\}$. ■

**Proof of Proposition 5:**

**(If)** I show that a matching that is not unit-stable violates at least one of size envy-freeness or 1-boundedness. Let $\mu$ be a matching with a unit blocking pair $(a, o) \in A \times O$. If $a \in D$, $a$ has a 2-unit claim to $o$ and $\mu$ is not 1-bounded. If $a \in S$ and $\sum_{a' \in \mu(o)} w_{a'} \neq q_o - 1$, then either $a \in S$ and $\sum_{a' \in \mu(o)} w_{a'} = q_o$ so $a$ envies at least one agent in $\mu(o)$, which means that $\mu$ is not size envy-free, or $\sum_{a' \in \mu(o)} w_{a'} \leq q_o - 2$, in which case $a$ has a 2-unit claim to $o$ and $\mu$ is not 1-bounded. If $a \in S$ and $\sum_{a' \in \widehat{\mu}_a(o)} w_{a'} \neq q_o - 1$, then either $\sum_{a' \in \widehat{\mu}_a(o)} w_{a'} = q_o$ and $(a, o)$ is not a blocking pair, a contradiction since a unit blocking pair is by definition a blocking pair, or $\sum_{a' \in \widehat{\mu}_a(o)} w_{a'} \leq q_o - 2$ and $a$ has a 2-unit claim to $o$, in which case $\mu$ is not 1-bounded. □

**(Only If)** I show that a matching that is either not size envy-free or not 1-bounded does not satisfy unit-stability. Let $\mu$ be a matching that is not size envy-free. Then, there exists two agents $a$ and $\widetilde{a}$ such that $a$ envies $\widetilde{a}$ and $w_a \leq w_{\widetilde{a}}$. For ease of notation, let $o \equiv \mu(\widetilde{a})$. The fact that $a$ envies $\widetilde{a}$ implies that $a$ has a $w_{\widetilde{a}}$-unit claim to $o$. Combined with the fact that $w_a \leq w_{\widetilde{a}}$, this means that $a$ has a $w_a$-unit claim to $o$ so $(a, o)$ is a blocking pair. By definition, $\widetilde{a} \in \mu(o) \setminus \widehat{\mu}_a(o)$ so $\sum_{a' \in \mu(o)} w_{a'} \neq \sum_{a' \in \widehat{\mu}_a(o)} w_{a'}$; therefore at least one of the two sums is not equal to $q_o - 1$ and $(a, o)$ is a unit-blocking pair so $\mu$ is not unit-stable.

Let $\mu$ be a matching that is not 1-bounded. Then, there exists a pair $(a, o) \in A \times O$ such that $a$ has a 2-unit claim to $o$. As $w_a \leq 2$, $(a, o)$ is a blocking pair. In addition, $\sum_{a' \in \widehat{\mu}_a(o)} w_{a'} \leq q_o - 2$ so $(a, o)$ is a unit blocking pair and $\mu$ is not unit-stable. ■

**Proof of Proposition 6:**

**(If)** I show that a matching that is not size-stable violates at least one of size envy-freeness or non-wastefulness. Let $\mu$ be a matching with a size blocking pair $(a, o) \in A \times O$. If $a \in S$, then $a$ has a 1-unit claim to $o$, meaning that at least one unit of $o$ is either assigned to a lower-priority agent, in which case $\mu$ is not size envy-free, or unassigned, in which case $\mu$ is wasteful. If $a \in D$, then $a$ has a 2-unit claim to $o$ and $2 \cdot |\widehat{\mu}_a(o) \cap D| + |\mu(o) \cap S| \leq q_o - 2$ so at least two units of $o$ are either assigned to a lower-priority double-unit agents, in which case $\mu$ is not size envy-free, or unassigned, in which case $\mu$ is wasteful. □

**(Only If)** I show that a matching that is either wasteful or not size envy-free does not satisfy size-stability. Let $\mu$ be a matching that is not size envy-free. Then, there exists two agents $a$ and $\widetilde{a}$ such that $a$ envies $\widetilde{a}$ and $w_a \leq w_{\widetilde{a}}$. For ease of notation, let $o \equiv \mu(\widetilde{a})$. The fact that $a$ envies $\widetilde{a}$ implies that $a$ has a $w_{\widetilde{a}}$-unit claim to $o$. Combined with the fact that $w_a \leq w_{\widetilde{a}}$, this means that $a$ has a $w_a$-unit claim to $o$ so $(a, o)$ is a blocking pair. If $a \in S$, this directly implies that $(a, o)$ is a size blocking pair and $\mu$ is not size-stable. If $a \in D$, then $w_a \leq w_{\widetilde{a}}$ implies $\widetilde{a} \in D$ and, since $a \rhd_o \widetilde{a}$, $|\mu(o) \cap D| \geq |\widehat{\mu}_a(o) \cap D| + 1$. Then,

$$2 \cdot |\widehat{\mu}_a(o) \cap D| + |\mu(o) \cap S| \leq 2 \cdot |\mu(o) \cap D| + |\mu(o) \cap S| - 2 = (\sum_{a' \in \mu(o)} w_{a'}) - 2 \leq q_o - 2,$$

where the last inequality comes from the fact that, by definition, matchings do not violate capacity constraints so $\sum_{a' \in \mu(o)} w_{a'} \leq q_o$. The above series of inequalities implies that $(a, o)$ is a size blocking pair so $\mu$ is not size-stable.

Let $\mu$ be a wasteful matching. Then, there exists a pair $(a, o) \in A \times O$ such that $o \succ_a \mu(a)$ and $\sum_{a' \in \mu(o)} w_{a'} \leq q_o - w_a$. $(a, o)$ is a blocking pair so $a \in S$ directly implies it is a size blocking pair and $\mu$ is not size-stable. If $a \in D$, then

$$2 \cdot |\widehat{\mu}_a(o) \cap D| + |\mu(o) \cap S| \leq 2 \cdot |\mu(o) \cap D| + |\mu(o) \cap S| = \sum_{a' \in \mu(o)} w_{a'} \leq q_o - w_a = q_o - 2$$

so $(a, o)$ is a size blocking pair and $\mu$ is not size-stable. ■

**Proof of Theorem 2:**

**($\boldsymbol{M^{UUS} \succeq M^{US}}$)** The result is trivial if $M^{US} = \varnothing$ so the proof concentrates on the case where $M^{US} \neq \varnothing$. By Propositions 2 and 5, unit-stability is characterized by two of the three axioms that characterize stability; therefore all stable matchings are unit-stable. Let $\nu$ be an undominated stable matching. Then, $\nu$ is unit-stable and, since domination is a partial order, $\nu$ is either an undominated unit-stable matching or it is dominated by one; therefore, there exists $\mu \in M^{UUS}$ such that $\mu \succeq \nu$. Consequently, $M^{UUS} \succeq M^{US}$ unless there exist $\mu \in M^{UUS}$ and $\nu \in M^{US}$ such that $\nu \succ \mu$. In that case, however, the fact that $\nu$ is unit-stable implies that $\mu \notin M^{UUS}$, a contradiction. □

**($\boldsymbol{M^{USS} \succeq M^{UUS}}$)** Denote by $M^{USE}$ the set of undominated size envy-free matchings, that is the set of matchings that are size envy-free and not dominated by any other size envy-free matching. Theorem 1 and the fact that domination is a partial order guarantee that this set is nonempty. As unit-stability is characterized by size envy-freeness and 1-boundedness (Proposition 5), all unit-stable matchings are size envy-free. By an analogous reasoning to the one above, this implies $M^{USE} \succeq M^{UUS}$. To complete the proof, it remains to show that $M^{USS} = M^{USE}$.

Towards a contradiction, suppose the existence of an undominated size envy-free matching $\mu$ that is not size-stable. Since size-stability is characterized by size envy-freeness and non-wastefulness (Proposition 6), $\mu$ is wasteful. Thus, there exists a blocking pair $(a, o)$ such that $o \succ_a \mu(o)$ and $\sum_{a' \in \mu(o)} w_{a'} \leq q_o - w_a$. Without loss of generality, let $a$ be the highest-priority agent in that situation. (That is, for all $\widetilde{a} \in A$ such that $o \succ_{\widetilde{a}} \mu(o)$ and $\sum_{a' \in \mu(o)} w_{a'} \leq q_o - w_{\widetilde{a}}$, $a \trianglerighteq \widetilde{a}$.) Let $\nu$ be a matching constructed as follows: $\nu(a) = o$ and $\nu(a') = \mu(a')$ for all $a' \neq a$. $\nu$ does not violate any capacity constraint since $\mu$ does not and $o$ has at least $w_a$ unassigned units at $\mu$. If $a \in D$, no agent envies $a$ since all agents who prefer $o$ to their own objects have a lower priority than $a$ for $o$. If $a \in S$, the same is true of single-unit agents. Therefore, $a$ is only envied by larger-size agents (if any) and, since $\mu$ is size envy-free, so is $\nu$. Since $\nu(a) = o \succ_a \mu(a)$ and $\nu(a') = \mu(a')$ for all $a' \neq a$, $\nu$ dominates $\mu$ so $\mu$ is dominated by another size envy-free matching, a contradiction. It follows that all undominated size envy-free matchings are size-stable. Combined with the fact that all size-stable matchings are size envy-free, this means that every undominated size envy-free matching is an undominated size-stable matching: $M^{USE} \subseteq M^{USS}$.

Again towards a contradiction, suppose this tme the existence of an undominated size-stable matching $\mu$ that is not an undominated size envy-free matching. As all size-stable matchings are size envy-free and domination is a partial order, $\mu$ is dominated by an undominated size envy-free matching $\nu$. However, because $M^{USE} \subseteq M^{USS}$, $\nu$ is size-stable so $\mu$ is not an undominated size-stable matching, a contradiction. Therefore, $M^{USS} \subseteq M^{USE}$, which means that $M^{USS} = M^{USE}$. ■

## Additional Results

I present below results that are mentioned but not formally stated in the main text, together with their proofs.

**Proposition 7.** *Unit-stable matchings can be made stable by removing at most one unit per object but matchings that can be made stable by removing at most one unit per object are not necessarily unit-stable.*

**Proof:** I first prove the first part of the statement. Let $\mu$ be a unit-stable matching. If it is stable, the proof is complete. Otherwise, let $o$ be an object involved in a blocking pair at $\mu$ and let $(s, o)$ be any of the blocking pairs that involve $o$. By definition, $s \in S$ and $\sum_{a' \in \mu(o)} w_{a'} = \sum_{a' \in \widehat{\mu}_s(o)} w_{a'} = q_o - 1$. Consider the modified market where one unit of $o$ is removed. Since one unit of $o$ was unassigned, $\mu$ is still a matching in this modified market and any blocking pair at $\mu$ in the modified market was already a blocking pair in the original market. In addition, $\sum_{a' \in \widehat{\mu}_s(o)} w_{a'} = q_o$ so $o$ is no longer involved in any blocking pair. A stable matching obtains by repeating this operation until none of the objects are involved in a blocking pair. □

I next show the second part of the statement by counterexample. Consider the unique size-stable matching in Example 1:

$$\mu^* \equiv \begin{pmatrix} s_1 & s_2 & d_1 \\ o_1 & o_2 & \emptyset \end{pmatrix}.$$

$\mu^*$ is not unit-stable since $d_1$ has a two-unit claim to $o_1$. However, removing the unassigned unit of $o_1$ yields a modified market where $q_{o_1} = 1$ and $\mu^*$ is stable. ■

**Proposition 8.** *d-domination is a partial order over the set of all matchings $M$ and set domination is a partial order over the set of all matching subsets $2^M$.*

**Proof:** I show that both d-domination and set domination are antisymmetric and transitive.

***d-Domination***

**(Antisymmetric)** Towards a contradiction, suppose the existence of two matchings $\mu_1, \mu_2 \in M$ such that $\mu_1 \succ^D \mu_2$ and $\mu_2 \succ^D \mu_1$. $\mu_1 \succ^D \mu_2$ implies that $\mu_1(a) \succ_a \mu_2(a)$ for some $a \in A$ so $\mu_2$ does not dominate $\mu_1$. Then, the fact that $\mu_2 \succ^D \mu_1$ implies that $\mu_2(d) \succ_d \mu_1(d)$ for some $d \in D$, which contradicts $\mu_1 \succ^D \mu_2$. □

**(Transitive)** Consider three matchings $\mu_1, \mu_2, \mu_3 \in M$ such that $\mu_1 \succ^D \mu_2$ and $\mu_2 \succ^D \mu_3$. One needs to show that $\mu_1 \succ^D \mu_3$. If $\mu_1 \succ \mu_2$ and $\mu_2 \succ \mu_3$, $\mu_1 \succ \mu_3$ since domination is a partial order (hence it is transitive). As domination implies d-domination, $\mu_1 \succ^D \mu_3$. Otherwise, the hypothesis implies that $\mu_1(d) \succeq_d \mu_2(d) \succeq_d \mu_3(d)$ for all $d \in D$. In addition, either $\mu_1$ does not dominate $\mu_2$, in which case $\mu_1 \succ^D \mu_2$ implies $\mu_1(d) \succ_d \mu_2(d)$ for some $d \in D$ or $\mu_2$ does not dominate $\mu_3$, in which case $\mu_2 \succ^D \mu_3$ implies $\mu_2(d) \succ_d \mu_3(d)$ for some $d \in D$. Combining either case with the fact that $\mu_1(d) \succeq_d \mu_2(d) \succeq_d \mu_3(d)$ for all $d \in D$ implies that $\mu_1(d) \succeq_d \mu_3(d)$ for all $d \in D$ and $\mu_1(d) \succ_d \mu_3(d)$ for some $d \in D$ so $\mu_1 \succ^D \mu_3$. □

***Set Domination***

**(Antisymmetric)** Towards a contradiction, suppose the existence of two sets of matchings $M_1, M_2 \in 2^M$ such that $M_1 \succ M_2$ and $M_2 \succ M_1$. Then, the following hold:

(a) For all $\mu_1 \in M_1$, there exists $\mu_2 \in M_2$ such that $\mu_2 \succeq \mu_1$,

(b) For all $\mu_1 \in M_1$, there does not $\mu_2 \in M_2$ such that $\mu_2 \succ \mu_1$,

(c) For all $\mu_2 \in M_2$, there exists $\mu_1 \in M_1$ such that $\mu_1 \succeq \mu_2$, and

(d) For all $\mu_2 \in M_2$, there does not $\mu_1 \in M_1$ such that $\mu_1 \succ \mu_2$.

Combining (a) and (b) implies that for all $\mu_1 \in M_1$, there exists $\mu_2 \in M_2$ such that $\mu_2 = \mu_1$, which in turn implies $M_1 \subseteq M_2$. Analogously, (c) and (d) imply $M_2 \subseteq M_1$ so $M_1 = M_2$, a contradiction. $\square$

**(Transitive)** Consider three sets of matchings $M_1, M_2, M_3 \in 2^M$ such that $M_1 \succ M_2$ and $M_2 \succ M_3$. One needs to show that $M_1 \succ M_3$. The hypothesis implies the following:

(a) For all $\mu_2 \in M_2$, there exists $\mu_1 \in M_1$ such that $\mu_1 \succeq \mu_2$,

(b) For all $\mu_3 \in M_3$, there exists $\mu_2 \in M_2$ such that $\mu_2 \succeq \mu_3$,

(c) For all $\mu_1 \in M_1$, there does not $\mu_2 \in M_2$ such that $\mu_2 \succ \mu_1$, and

(d) For all $\mu_2 \in M_2$, there does not $\mu_3 \in M_3$ such that $\mu_3 \succ \mu_2$.

As domination is transitive, combining (a) and (b) directly implies that, for all $\mu_3 \in M_3$, there exists $\mu_1 \in M_1$ such that $\mu_1 \succeq \mu_3$. Then $M_1 \succ M_3$ unless there exists $\mu_1 \in M_1$ and $\mu_3 \in M_3$ such that $\mu_3 \succ \mu_1$. In that case, by (b) there exists $\mu_2 \in M_2$ such that $\mu_2 \succeq \mu_3 \succ \mu_1$, which, as domination is transitive, contradicts (c). $\blacksquare$

**Proposition 9.** *An undominated stable matching may be dominated by a unit-stable matching and by a size-stable matching.*

**Proof:** I first provide an example to show the first part of the statement: an undominated stable matching may be dominated by a unit-stable matching.

**Example 6** (Unit-stable dominates USM)**.** Let $S = \{s_1, s_2\}$, $D = \{d_1, d_2, d_3\}$, $O = \{o_1, o_2, o_3, o_4, \emptyset\}$, and $(q_{o_1}, q_{o_2}, q_{o_3}, q_{o_4}, q_\emptyset) = (2, 2, 2, 2, 8)$. The preferences and priorities are given below.

| $s_1$ | $s_2$ | $d_1$ | $d_2$ | $d_3$ |
|---|---|---|---|---|
| $o_4$ | $o_2$ | $o_1$ | $o_3$ | $o_2$ |
| $o_2$ | $o_3$ | $o_2$ | $o_4$ | $\emptyset$ |
| $o_1$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\vdots$ |
| $\emptyset$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| $o_3$ | | | | |

| $o_1$ (2) | $o_2$ (2) | $o_3$ (2) | $o_4$ (2) | $\emptyset$ (8) |
|---|---|---|---|---|
| $s_1$ | $d_1$ | $s_2$ | $d_2$ | |
| $d_1$ | $s_1$ | $d_2$ | $s_1$ | |
| | $d_3$ | | | $\vdots$ |
| $\vdots$ | $s_2$ | $\vdots$ | $\vdots$ | |
| | $d_2$ | | | |

**Lemma 2.** *There are exactly two unit-stable matchings in Example 6:*

$$\mu = \begin{pmatrix} s_1 & s_2 & d_1 & d_2 & d_3 \\ o_1 & o_3 & o_2 & o_4 & \emptyset \end{pmatrix} \quad \textit{and} \quad \nu = \begin{pmatrix} s_1 & s_2 & d_1 & d_2 & d_3 \\ o_2 & o_3 & o_1 & o_4 & \emptyset \end{pmatrix}.$$

It is easy to verify that $\mu$ does not have any blocking pair, hence it is stable. On the other hand, $\nu$ has one blocking pair: $(s_2, o_2)$. However, $s_2 \in S$ and

$$\sum_{a' \in \nu(o_2)} w_{a'} = \sum_{a' \in \widehat{\nu}_{s_2}(o)} w_{a'} = w_{s_1} = 1 = q_{o_2} - 1$$

so $(s_2, o_2)$ is not a unit blocking pair and $\nu$ is unit-stable. I complete the proof of Lemma 2 separately below by showing that there are no other unit-stable matchings.

Notice that $\nu$ dominates $\mu$ since $s_1$ and $d_1$ are better-off at $\nu$ and all other agents are matched to the same object. Therefore, $\mu$ is an undominated stable matching (since it is the unique stable matching by Lemma 2) and is dominated by $\nu$, a unit-stable matching. □

I next provide an example to show the second part of the statement: an undominated stable matching may be dominated by a size-stable matching.

**Example 7** (Size-stable dominates USM)**.** Let $S = \{s_1, s_2, s_3\}$, $D = \{d_1\}$, $O = \{o_1, o_2, \emptyset\}$, and $(q_{o_1}, q_{o_2}, q_\emptyset) = (2, 1, 5)$. The preferences and priorities are given below.

| $s_1$ | $s_2$ | $s_3$ | $d_1$ |
|---|---|---|---|
| $o_1$ | $o_2$ | $o_1$ | $o_1$ |
| $o_2$ | $o_1$ | $o_2$ | $\emptyset$ |
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $o_2$ |

| $o_1$ (2) | $o_2$ (1) | $\emptyset$ (5) |
|---|---|---|
| $s_2$ | $s_1$ | |
| $d_1$ | $s_3$ | $\vdots$ |
| $s_1$ | $s_2$ | |
| $s_3$ | $d_1$ | |

**Lemma 3.** *There are exactly two size-stable matchings in Example 7:*

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & d_1 \\ o_1 & o_1 & o_2 & \emptyset \end{pmatrix} \quad \textit{and} \quad \nu = \begin{pmatrix} s_1 & s_2 & s_3 & d_1 \\ o_1 & o_2 & o_1 & \emptyset \end{pmatrix}.$$

It is easy to verify that $\mu$ does not have any blocking pair, hence it is stable. On the other hand, $\nu$ has one blocking pair: $(d_1, o_1)$. However, $d_1 \in D$ and

$$2 \cdot |\widehat{\nu}_{d_1}(o_1) \cap D| + |\nu(o_1) \cap S| = 2 \cdot |\varnothing| + |\{s_1, s_3\}| = 2 \cdot 0 + 2 = 2 = q_{o_1} > q_{o_1} - 2$$

so $(d_1, o_1)$ is not a size blocking pair and $\nu$ is size-stable. I complete the proof of Lemma 3 separately below by showing that there are no other size-stable matchings.

Notice that $\nu$ dominates $\mu$ since $s_2$ and $s_3$ are better-off at $\nu$ and all other agents are matched to the same object. Therefore, $\mu$ is an undominated stable matching (since it is the unique stable matching by Lemma 3) and is dominated by $\nu$, a size-stable matching. ■

**Proof of Lemma 2:** Let $\xi$ be a unit-stable matching. First of all, notice that if $o_1 = \mu(s_1) \succ_{s_1} \xi(s_1)$, then $s_1$ has a 2-unit claim to $o_1$ and $\xi$ is not unit-stable so $\xi(s_1) \succeq_{s_1} \mu(s_1) = o_1$. An analogous reasoning holds for the other agents so $\xi \succeq \mu$. Then, $s_1$ is matched to one of $o_4$, $o_2$, or $o_1$. I consider each case separately.

If $\xi(s_1) = o_4$, then $d_2$ cannot be matched to $o_4$ so $\xi(d_2) = o_3$, which similarly implies $\xi(s_2) = o_2$. In addition, $\xi(s_1) = o_4$ implies $\xi(d_1) = o_1$ as, otherwise, $d_1$ has a 2-unit claim to $o_1$. $d_3$ cannot be matched to $o_2$ since $s_2$ is, so $\xi(d_3) = \emptyset$. The matching obtained is

$$\begin{pmatrix} s_1 & s_2 & d_1 & d_2 & d_3 \\ o_4 & o_2 & o_1 & o_3 & \emptyset \end{pmatrix},$$

which is not unit-stable since $d_3$ has a 2-unit claim to $o_2$.

If $\xi(s_1) = o_2$, $d_1$ and $d_3$ cannot be matched to $o_2$ so $\xi(d_1) = o_1$ and $\xi(d_3) = \emptyset$. In addition, $\xi(d_2) = o_4$ as, otherwise, $s_1$ has a 2-unit claim to $o_4$. In turn, $d_2$ has a 2-unit claim to $o_3$ unless $\xi(s_2) = o_3$. The matching obtained is $\nu$.

If $\xi(s_1) = o_1$, then $\xi(d_1) = o_2$ as $s_1$ and $d_1$ cannot be both matched to $o_1$. In turn, no agent other than $d_1$ is matched to $o_2$ so $\xi(s_2) = o_3$ and $\xi(d_3) = \emptyset$. Then, $d_2$ is not matched to $o_3$ so $\xi(d_2) = o_4$ and $\mu$ obtains.

It follows that no matching other than $\mu$ and $\nu$ is unit-stable, which completes the proof since $\mu$ and $\nu$ were shown to be unit-stable in the proof of Proposition 9. ■

**Proof of Lemma 3:** Let $\xi$ be a size-stable matching and suppose first that $\xi(d_1) = o_1$. Then, no other agent is matched to $o_1$ and $\xi(s_1) = o_2$ as, otherwise, $(s_1, o_2)$ is a size blocking pair. Since all units of $o_1$ and $o_2$ are assigned, $\xi(s_2) = \xi(s_3) = \emptyset$, which yields

$$\begin{pmatrix} s_1 & s_2 & s_3 & d_1 \\ o_2 & \emptyset & \emptyset & o_1 \end{pmatrix}.$$

That matching is not size-stable because $s_2$ envies $d_1$. If $\xi(d_1) = o_2$, $\xi$ is wasteful since $\emptyset \succ_{d_1} o_2$; therefore, $\xi(d_1) = \emptyset$ in all size-stable matchings. $o_1$ has a quota of two units and is $s_1$'s first preference. Since $d_1$ is not matched to $o_1$, $s_1$ has a 1-unit claim to $o_1$ (hence $(s_1, o_1)$ is a size blocking pair) unless $\xi(s_1) = o_1$. Either one of $\xi(s_2) = \emptyset$ or $\xi(s_3) = \emptyset$ creates a size blocking pair since the null is the last preference of both $s_2$ and $s_3$ and one unit of each of $o_1$ and $o_2$ remains to be assigned. Thus, there are two possibilities: $\xi(s_2) = o_1$ and $\xi(s_3) = o_2$, which yields $\mu$, and $\xi(s_2) = o_2$ and $\xi(s_3) = o_1$, which yields $\nu$. It follows that no matching other than $\mu$ and $\nu$ is size-stable, which completes the proof since $\mu$ and $\nu$ were shown to be size-stable in the proof of Proposition 9. ■